Financial Hedging and Optimal Currency of Invoicing

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Abstract

I develop a theory of the optimal currency choice for invoicing goods for international trade in the presence of imperfect financial hedging of currency risk. I demonstrate that the classic irrelevance result—that the cost of financial hedging does not impact the choice of currency invoicing—rests on the assumption that sellers set prices ex-ante and fulfill any order size ex-post. I show that when quantities are also sticky, in the sense that the order quantity is in part pre-specified, then financial hedging affects the optimal currency of invoicing choice. My theory incorporates the cost of FX financial hedging into the classic theory of optimal currency choice, which relies on real hedging. I show that the optimal currency choice takes into account the relative ability of buyers and sellers to bear exchange rate risk. This financial hedging channel generates feedback between macroprudential policies, such as capital controls, and the optimal currency of invoicing. I highlight a "dollarization dilemma": capital control policies that aim to reduce dollar borrowing are partially offset by an endogenous substitution into dollar-invoiced trade, which amplifies the local economy's exposure to the dollar's movements.

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Exchange rate fluctuations play a crucial role in redistributing demand between domestic and foreign goods. In the presence of sticky prices, the direction and magnitude of these effects depend on the currency in which a good is invoiced. Indeed, the study of the optimal currency of invoicing choice by firms is at the core of international macroeconomics and finance. However, classic theoretical frameworks do not offer any role for financial hedging in determining the currency choice of trade invoicing, which is instead pinned down exclusively by a real hedging channel. I provide a theory of the optimal currency of invoicing that unifies the theories of real and financial hedging. I show that this theory is important for understanding firm behavior, the dominance of the dollar as a currency of invoicing, and alters the normative prescription for optimal capital controls.

When prices are set in advance, the currency of invoicing affects the prices that the buyer faces, as well as the value of the sales to the seller. For example, if the unit price is set in the currency of the seller (producer currency pricing), a producer currency depreciation makes the goods cheaper for the buyer, who might consequently increase their demand. On the other hand, if the price is set in the currency of the buyer (local currency pricing), a producer currency depreciation does not affect the buyer's demand, but it does lower the producer currency value of a given amount of sold goods. The classic theory of the optimal choice of currency invoicing sets the price in the currency that best helps the seller to replicate the price they would have set under fully flexible prices. In the theory of real hedging, this flexible price reflects the real risks of international trade, such as uncertainty over realized marginal costs and output demand. This theory offers no role for the ability to hedge financial risks associated with fluctuations in foreign exchange rates (FX), leading to a classic result of the *irrelevance of financial hedging* for trade invoicing.

I show that this classic irrelevance result rests on the assumption that sellers set prices in advance and fulfill any order size ex-post. I refer to this setup as sticky prices and flexible quantities. I show that if quantities in trade are also in part determined ex-ante, then the classic irrelevance result breaks down and a new financial motive for currency of invoicing choice arises. Under sticky quantities and sticky prices, trade invoicing is also used to hedge FX financial risk. Buyers and sellers, in the presence of imperfect FX hedging markets, might differ in their willingness to bear this risk. A buyer, for example, might be willing to purchase the good with the price set in the producer currency, thus insuring the producer, whenever the buyer has a higher risk-bearing capacity. In this case, the buyer uses the currency of invoicing to extract a lower price for the sale as an insurance premium.

This financial hedging mechanism is often cited in firm invoicing decisions. For example, the US supermarket Trader Joe's explains how it imports Dijon mustard in the euro to improve FX risk sharing: "part of the intensive involvement in the buying process is to reduce risk for our vendors. We pay for that mustard in euros, their local currency now. We assume the risk of that currency fluctuation. . . [so that] our costs go down" (Miller and Sloan, 2021). Invoicing Dijon mustard in the euro reduces the risk for the vendors and lowers the average price for Trader Joe's.

My result for the optimal currency of invoicing unifies the financial hedging motive with the classic real hedging motive. While my main focus is on applied theory, I show that quantitative calibrations with standard parameters from the literature point towards an important role for financial hedging motives. This is especially the case in explaining the prevalence of dollar trade invoicing when neither the buyer nor the seller is US based (dominant currency pricing). I illustrate the importance of the financial hedging motive with a natural experiment in which China and Korea established a bilateral market for the RMB-KRW exchange rate. This bilateral market reduced FX transaction costs and increased the willingness of Korean exporters to bear the risk of fluctuations in the RMB exchange rate. My theory rationalizes the resulting transition from dollar to renminbi invoicing as a result of the financial hedging motive.

Finally, I integrate my theory into a small open economy New Keynesian model to study its policy implications in general equilibrium. I show that capital controls on dollar borrowing, which are often used by governments to reduce excessive dollar debt, can backfire by amplifying dollar trade. This induces a "dollarization dilemma" for the design of optimal policy. When trade quantities are sticky, a tax on dollar financial borrowing causes local agents to substitute away from debt contracts toward buying goods in the dollar, which creates a synthetic dollar liability. The substitution between financial dollarization and trade dollarization is a form of regulatory arbitrage with real consequences. The substitution can more than offset the benefits of dollar capital controls, since it leads to an increase in dominant currency pricing: which is particularly destabilizing for countries that tend to rely on imports when the dollar appreciates. My theory links FX financial markets and aggregate trade invoicing patterns, which highlights an important macroeconomic consideration for the regulation of financial markets.

This paper builds on the seminal work of Engel (2006). The classic analysis of real hedging considers a seller exporting a good and choosing ex-ante whether to price the good in its home currency or that of the destination market. Ex-post, the seller commits to deliver whatever quantity is demanded at the point of sale. The crux of the analysis lies in demonstrating how the choice of currency invoicing can approximate a flexible, state-contingent pricing mechanism. For instance, a seller who aims to dynamically adjust prices based on a desired markup over marginal cost can approximate this by fixing prices in the currency whose exchange rate most closely covaries with these fluctuations.

One key assumption of the classic theory of real hedging is that prices are set ex-ante and quantities are ex-post determined. A voluminous literature builds on the classic theory and evaluates its predictions in the data, discusses its significance in macro models, and extends its insights to currency choices beyond those of the seller and buyer (e.g. dominant currency pricing). Throughout, the maintained result is that the external financial hedging positions of exporters and importers do not affect the invoicing currency choice in trade.

In practice, quantities are often prespecified in international goods trade. Sellers often commit to the number of goods because they have to ship goods overseas. Most obviously, sellers do not commit to fulfill arbitrarily large order sizes at a given price, and my theory shows that this abstraction in the existing work is not innocuous for the results and policy conclusions. When quantities are fixed, the seller factors in the cost of foreign exchange financial hedging in their choice of currency of invoicing.

Even when quantities are sticky, my theory demonstrates that the cost and availability of external financial hedging matters only if capital markets are also imperfect, precisely because this is a theory of risk sharing through the trade contract. For example, Trader Joe's agrees to invoice in the euro to share euro-dollar FX risk only because the Dijon producer cannot self-insure using a euro-dollar forward at the same price. If both parties could access the same forward market at the same price, they could engage in side bets to avoid distorting the trade-invoicing decision. Imperfect capital markets cause differential valuations of the risk of foreign exchange between the producer and the importer.

Although foreign exchange (FX) markets are among the most liquid in the world, they operate with significant frictions. Even for the most actively traded currencies, covered interest parity (CIP) deviations are widespread (Du and Schreger, 2016; Du, Tepper and Verdelhan, 2018). These deviations provide direct evidence of imperfect FX hedging: they measure the gap between the fair-value forward rate implied by no-arbitrage conditions and the observed market forward rate. Because FX forwards are traded over the counter, financial intermediaries possess market power and charge substantial spreads—on average 67 basis points for single-banked clients (Hau et al., 2021). These frictions are particularly acute in emerging market economies, where dominant currency pricing is prevalent and capital controls or nonconvertibility policies further constrain forward market liquidity (Cerutti, Obstfeld and Zhou, 2021). The combination of these forces generates asymmetric valuations of exchange rate risk between buyers and sellers, amplifying the role of financial hedging in trade invoicing decisions.

I apply the model to characterize how different pricing regions emerge and how buyer-seller heterogeneity shapes the optimal currency of trade invoicing. To parallel my event study—which examines the shift from dollar to renminbi invoicing in Korean exports to

China—I calibrate a setting in which a Korean exporter sells to a small Chinese importer and chooses to invoice in the won, renminbi, or dollar. The pricing regions reveal how the financial hedging motive alters the set of buyer-seller pairs that find it optimal to invoice in the dominant or local currency. In particular, when the Chinese importer cannot financially hedge its FX exposure, Korean exporters optimally invoice in the dollar or renminbi to assume the risk of exchange rate fluctuations. In my calibration, I find that the introduction of the 2014 bilateral RMB-KRW exchange rate market generates a sizable increase in renminbi invoicing, consistent with the empirical evidence.

To deliver these insights with the simplicity and transparency of Engel (2006), in Section 1, I first analyze trade invoicing using a reduced-form quantity restriction, two currencies, and marginal utilities of wealth modeled with exogenous stochastic discount factors. A fraction of trade is fixed in advance and the remaining fraction is determined at the time of sale. This basic environment helps me characterize the main results and intuition. In Section 2, I apply my theory to study how heterogeneous buyer-seller pairs would choose different currencies (the producer, local, or dominant currency) depending on their characteristics. I document the empirical plausibility of my mechanism in Section 3 and characterize its macroeconomic and financial implications in Section 4. In Section 5, I generalize the baseline model and develop the renegotiation-proof contracting solution. I find that the financial hedging mechanism is present in trade contracts except when there is an absence of commitment because of the threat of renegotiation. Quantities do not have to be fixed across states: for example, the optimal trade contract may specify fixed quantities in some states but allow adjustment when the threat of renegotiation is binding. I show how the optimal contract resembles the reduced-form problem in earlier sections.

Literature Review.—The currency of invoicing for internationally traded goods has been of central importance in macroeconomic models and policy. In the presence of sticky prices, exchange rates determine relative international prices and influence consumer and producer decisions. This effect is central to Keynesian analysis (Keynes, 1923; Dornbusch, 1976; Obstfeld and Rogoff, 1995). Engel (2006) studied how an exporter optimally chooses the currency of invoicing to retain monopoly pricing power. A large literature has built on this theory of real hedging by studying producer, local, and dominant currency pricing (Bacchetta and van Wincoop, 2005; Gopinath, Itskhoki and Rigobon, 2010; Burstein and Gopinath, 2014; Corsetti and Pesenti, 2015; Goldberg and Tille, 2016; Devereux, Dong and Tomlin, 2017; Gopinath et al., 2020; Amiti, Itskhoki and Konings, 2022; Gopinath and Itskhoki, 2022; Mukhin, 2022; Benguria and Wagner, 2024; De Gregorio et al., 2024).

A separate literature emphasizes the financial hedging mechanism. Classic theories point to liquidity (Krugman, 1980; Matsuyama, Kiyotaki and Matsui, 1993), financing costs

(Gopinath and Stein, 2021; Chahrour and Valchev, 2022; Coppola, Krishnamurthy and Xu, 2023), and payment risk (Doepke and Schneider, 2017; Drenik, Kirpalani and Perez, 2022) as the source of currency denomination decisions. These insights relate to a corporate finance literature on financial hedging, which discusses a potential set of deviations from Modigliani and Miller (1958)'s capital structural irrelevance result that causes financial hedging to increase a firm's financial value (Stulz, 1984; Smith and Stulz, 1985; DeMarzo and Duffie, 1991, 1995; Froot, Scharfstein and Stein, 1993; Rampini and Viswanathan, 2010). Although the literature has noted the role of prespecified quantities in generating FX hedging motives, there is no general insight characterizing when and how this relates to the optimal choice of currency invoicing. My paper also contributes to a growing empirical literature that documents the empirical relevance of financial hedging in trade invoicing decisions (Döhring, 2008; Ito et al., 2018; Licandro and Mello, 2019; Alfaro, Calani and Varela, 2021; Lyonnet, Martin and Mejean, 2022; Son, 2023; Benguria and Novy, 2025).

My contribution is to show that the financial hedging mechanism affects trade invoicing when quantities are sticky and FX hedging is imperfect. Sticky quantities are fundamental to understanding the effect of exchange rate fluctuations on trade balances (Magee, 1973). The majority of US trade contracts fix quantities (Gopinath and Rigobon, 2008). Trade contracts regularly prespecify quantities to clarify their collateral value (Amiti and Weinstein, 2011). This enables the contract to be used in trade credit, facilitating financing and legal enforcement (Antràs and Foley, 2015). More generally, trade agreements are often imperfectly contractable, leading to optimally sticky quantity menus (Corrao, Flynn and Sastry, 2023; Flynn, Nikolakoudis and Sastry, 2024). By prespecifying quantities, the trade contract attenuates the expenditure-switching role of exchange rates in trade (Fitzgerald, Yedid-Levi and Haller, 2025). Instead of flexibly adjusting demand for foreign-denominated goods, a buyer must commit to purchasing a prespecified amount as written in the contract. This dampens the immediate expenditure-switching effects of exchange rates, which also helps to reconcile the dynamic point estimates of the expenditure-switching effect of exchange rates found in Berman, Martin and Mayer (2012); Devereux, Dong and Tomlin (2017); Auer et al. (2019); Barbiero (2021); and Amiti, Itskhoki and Konings (2022).

Even among major currencies, FX hedging is often subject to deviations in the no-arbitrage covered interest parity relation (Du and Schreger, 2016; Du, Tepper and Verdelhan, 2018; Olivier et al., 2025). These deviations reflect the supply and demand for FX forward contracts, and indirectly, the costs of bank regulation (Gabaix and Maggiori, 2015; Bahaj and Reis, 2022; Maggiori, 2022; Moskowitz et al., 2024*a*; Siriwardane, Sunderam and Wallen, 2025). Transaction costs and over-the-counter market segmentation further restrict access to forward markets across buyers and sellers, as single-banked clients face substantially higher hedging costs (Hau et al., 2021; Ranaldo and Somogyi, 2021). These imperfections give rise to the vehicle-currency role of the U.S. dollar in FX transactions (Somogyi, 2022), a pattern amplified in emerging markets where capital controls restrict FX risk sharing (Cerutti, Obstfeld and Zhou, 2021). Maggiori, Neiman and Schreger (2020) document that heterogeneous hedging frictions affect which assets mutual funds buy.

Finally, the broader implication of this paper is to revisit some of the conclusions in the New Keynesian literature on optimal macro prudential policy. Classic papers by Obstfeld and Rogoff (1995) and Clarida, Galí and Gertler (2001, 2002), as well as Galí and Monacelli (2005), assume that sellers export in their home currency. More recent literature connects this to the presence of dominant currency pricing, often in the form of invoicing in the US dollar (Devereux, Shi and Xu, 2007; Goldberg and Tille, 2009; Farhi and Werning, 2014, 2016; Basu et al., 2020, 2023; Bocola and Lorenzoni, 2020; Bianchi and Lorenzoni, 2022; Corsetti, Dedola and Leduc, 2023; Egorov and Mukhin, 2023; Fukui, Nakamura and Steinsson, 2025; McLeay and Tenreyro, 2025). In these papers, the currency invoicing choice is set exogenously and does not react to the economic environment. I relax this assumption and reanalyze the classic policy prescriptions for taxes on home and foreign-currency debt. My theory of trade invoicing accounts for the effects of capital controls on pricing patterns through Euler equation wedges, which are a key theoretical object in the analysis of FX markets in international macroeconomics (Gabaix and Maggiori, 2015; Farhi and Werning, 2016; Ottonello, 2021). Accounting for the effect on trade invoicing changes standard prescriptions of capital controls, as a government must choose between fighting trade dollarization and financial dollarization.

1 Baseline Model

This section emphasizes the basic mechanism. I focus on a two-currency case in which quantities are sticky because a fraction of the total amount sold is set in advance. The setting clarifies when and how financial hedging affects currency invoicing decisions. To deliver the result, I assume uniform pricing and rule out pricing-to-market effects. I generalize the main results in a contracting framework of Section 5 to show that neither assumption—of fixed quantities or uniform pricing—affects the main results.

1.1 Setting

The model is static. There is uncertainty over the state of the world $x \in X$, where x is a finite real random vector that occurs with probability $\mu(x)$. All agents take x as given.

There is a producer and a local currency with an exchange rate normalized to 1 in the first period and denoted by S in the second period. The producer currency is the numeraire for all payoffs and prices. The appreciation of the exchange rate is denoted by a small $s = S - 1$, so that a $s = 2\%$ implies that the producer currency depreciated two percent relative to the local currency. The producer and local currency gross risk-free interest rates are denoted by R and R^*. Forward contracts also exist, specifying a predetermined exchange rate F per unit of local currency to be converted into the producer currency in the second period.

There is a seller i and a market j. The seller is an owner of an exporting firm. The market is a measure of identical buyers, each indexed by $m \in [0, 1]$. The seller i and the buyers of market j have exogenous stochastic discount factors M^i and M^j (in units of the producer currency), respectively. I use these discount factors to model each agent's marginal utility of wealth in reduced form, so that an increase in M corresponds to an increase in the marginal utility. The covariance of the marginal utility of wealth with the exchange rate generates a currency risk premium. Following the literature, e.g. Engel (2006), I assume that these SDFs are exogenous, with the implication that the seller's currency choice does not affect the buyer and seller's marginal utility of wealth. This is plausible, for example, when the transaction is a small part of the buyer and seller's overall economic activity. Differences in the marginal utility of wealth $M^i \neq M^j$ capture imperfect risk sharing, since it implies that the buyer-seller pair has differential valuations of risk in equilibrium.

I assume that the seller sets the terms of the trade contract. Formally, a trade contract is a unit price $P(x)$ (in units of the producer currency) and quantity $Q(x)$ schedule that are functions of the underlying state x. The seller sets the schedule to maximize the discounted profits. Given a realized state x, the seller's profit function π is given by

$$\pi\left(P\left(x\right), Q\left(x\right), x\right).$$

This profit function is in units of the producer currency. It is normalized to an outside option of 0 and satisfies the following restrictions $\partial_P \pi \geq 0$, $\partial_Q \pi > 0$, and $\pi(\cdot, 0, \cdot) = 0$. I assume it is analytic. These restrictions are satisfied by most profit functions, such as $\pi(P, Q, x) = (P - C(x)) Q$, provided that the price exceeds the marginal cost $P \geq C(x)$.

The seller, all else equal, prefers to sell more goods at higher prices. However, the trade contract must also provide weakly positive value to each buyer. Each buyer m in market j derives value from trade equal to

$$V\left(Q^m\left(x\right), x\right) Q^m\left(x\right) - P\left(x\right) Q^m\left(x\right).$$

This represents the total value of the good for the buyer, VQ, less the total payment made

PQ, in units of producer currency.[1] The value of trade is also normalized to an outside option of 0 so that the participation constraint binds when the buyer's unit value equals unit price $V(Q^m(x), x) = P(x)$. I assume that the function V decreases in the quantity of trade Q^m and is analytic. An example of V is the standard Dixit-Stiglitz inverse demand curve $V(Q^m, x) = \mathcal{P}(Q^m/\mathcal{Q})^{-1/\sigma}$, where the ideal price $\mathcal{P}$ and the quantity $\mathcal{Q}$ are random variables part of x and $\sigma > 1$ is a constant elasticity of demand.

I now layer on two assumptions regarding the set of feasible price and quantity schedules. The first is standard from the literature (Engel, 2006).

Assumption 1 (Price Schedule)**.** *Prices are fixed in advance but denominated in a share of the local currency $\beta/P_0 \in [0,1]$*

$$P(s) = P_0 + \beta s \qquad \forall x \in X. \tag{1}$$

This states that the realized price is uniform across buyers and sticky in a currency. For example, if trade is entirely invoiced in the producer currency, then $\beta = 0$ so that prices do not change across states $P = P_0$. If instead trade is entirely invoiced in local currency, then $\beta = P_0$, so that the producer currency price changes depending on the realized exchange rate $P = P_0 S$.[2] In absence of this restriction, a firm would set unit prices ex-post so that $P(x)$ depends on the whole vector of x, and not just the exchange rate s.

The second assumption, specific to this paper, is the quantity schedule. I assume that a fraction of quantities are **sticky**. To highlight the financial hedging mechanism, in this section I assume that quantities are sticky because a fraction is fixed in advance. Otherwise, they are said to be **flexible**.

Assumption 2 (Quantity Schedule)**.** *A fraction $m \in [0, \delta]$ of buyers in market j fix quantities ex-ante Q^m_{ea} while the remaining fraction $m \in [\delta, 1]$ determines quantities ex-post $Q^m_{ep}(x)$, so that the total demand is*

$$Q(x) = \delta Q^m_{ea} + (1 - \delta) Q^m_{ep}(x) \qquad \forall x \in X.$$

This quantity schedule reflects the distinction between standalone contracts and master agreements with ex-post quantity determination. In a standalone contract, price and quantity are fixed ex-ante, making the agreement sufficiently definite to be legally binding.[3] By

[1] In Section 5, I show that the results do not depend on the functional form of the buyer's value function.

[2] In Engel (2006), the prices are set to be log-linear in exchange rates. Identical formulas and insights can be derived under a log-linear specification—however, in discrete-time asset pricing, it is standard to express exchange rate exposure linearly, removing the need to approximate Euler equations.

[3] See the United Nations Convention on Contracts for the International Sale of Goods, Article 14.

contrast, master agreements (e.g., master service, supply, or vendor agreements) establish long-term trading relationships in which the seller posts a menu of prices and the buyer issues orders as demand materializes. This mirrors the assumption of ex-post demand in the model. In Section 3.1, I discuss how the share of fixed-quantity commitments, denoted by δ, varies with the microstructure of the market for the seller's good.

By setting the price and quantity, the seller internalizes the buyer's demand curve and acts as a monopolist. Hence, the seller holds the buyer to their outside option. Fix the price schedule $P(s)$. For the fraction of buyers who determine quantities ex-post, this is formalized by the state-by-state condition

$$V\left(Q_{ep}^{m}(x), x\right) Q_{ep}^{m}(x) - P(s)\, Q_{ep}^{m}(x) = 0 \qquad \forall m \in [\delta, 1], x \in X.$$

For these buyers, the seller optimally sets $Q_{ep}^{m}(x) = V_{(x)}^{-1}(P(s))$ as the inverse demand curve holding x fixed. Using the earlier Dixit-Stiglitz demand example, ex-post demand would follow the formula $Q_{ep}^{m}(x) = \mathcal{Q}\left(P(s)/\mathcal{P}\right)^{-\sigma}$, which decreases in the realized price $P(s)$ and increases with the buyer's ideal price $\mathcal{P}$ and quantity $\mathcal{Q}$, both elements of the random vector x.

When quantities are fixed ex-ante, the buyer's surplus is determined in expectation and discounted by their stochastic discount factor M^j,

$$\mathbb{E}\left[M^j \left(V\left(Q_{ea}^{m}, x\right) Q_{ea}^{m} - P(s)\, Q_{ea}^{m}\right)\right] = 0 \qquad \forall m \in [\delta, 1].$$

The seller chooses $Q_{ea}^{m} \geq 0$ that satisfies this condition. This equation implicitly defines the optimal quantity $Q_{ea}^{m} := \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right)$. Unlike ex-post demand $Q_{ep}^{m}(x) = V_{(x)}^{-1}(P(s))$, it is downward sloping in the expected discounted price $\mathbb{E}\left[M^j P\right]$. For the total quantity demanded by the destination market, the monopolist faces the demand curve

$$Q(x) = \underbrace{\delta \bar{V}^{-1}\left(\mathbb{E}\left[M^j P(s)\right]\right)}_{\text{Ex Ante Demand}} + \underbrace{(1-\delta)\, V_{(x)}^{-1}(P(s))}_{\text{Ex Post Demand}} \qquad \forall x \in X. \tag{2}$$

Substituting in the seller's optimal quantity schedule, the **seller's problem** is to choose a price level P_0 and currency denomination β, taking as given the measure over states $\mu : X \mapsto \mathbb{R}_+$, to maximize discounted profits subject to the price and quantity schedule

$$\max_{P_0, \beta} \mathbb{E}\left[M^i \pi\left(P(s), Q(x), x\right)\right] \qquad \text{(Seller's Problem)}$$

$$\text{s.t. Equations (1) and (2) hold.}$$

Remark. The baseline model assumes a monopoly pricing problem with partially fixed quantities to emphasize classic theories of currency invoicing. The assumption of monopoly pricing and flexible quantities $\delta = 0$ is central to the real hedging literature, which relies on a standard markup motive. Instead, the financial hedging literature fixes quantities $\delta = 1$, but does not rely on monopoly pricing. Section 5 endogenizes commitment through quantities and shows that the results do not depend on uniform pricing.

1.2 Flexible Quantities

I start by analyzing currency invoicing when quantities are flexible $\delta = 0$. This benchmark mirrors the analysis of real hedging in Engel (2006). I therefore keep the exposition of this classic result streamlined.

I start by defining the concept of financial risk, which is central to my analysis.

Definition 1. The **financial risk** generated by prices π_P and shocks π_x is given by the total derivative onto firm profits

$$\pi_P := \partial_P \pi + \partial_Q \pi \cdot \partial_P Q \qquad \pi_x := \partial_x \pi + \partial_Q \pi \cdot \partial_x Q.$$

When prices vary, they generate financial risk through two mechanisms. First, a valuation effect $\partial_P \pi$: an increase in prices leads to a direct increase in the total value of profits, holding quantities fixed. Second, a quantity adjustment $\partial_P Q$ which occurs because quantities are decreasing in the realized price. The "real risk" $\partial_Q \pi$ of international trade is related to uncertainty about the cost and demand for the traded good, which affects the firm's profit margin and the ability to charge markups. Consequently, the real risk scales with uncertainty over quantities: When quantities are entirely fixed ex-ante $\delta = 1$, real risk disappears $\partial_P Q = 0$. Both total derivatives, denoted by the subscript of the profit function, will be used for deriving the optimal currency of invoicing.

Flexible Price Solution—In the special case of jointly flexible prices and quantities, the posted price is state-contingent, and quantities are determined ex-post. In other words, it is as if the seller, in each state of the world $x \in X$, chooses the optimal price $P(x)$ (in contrast to the sticky price $P(s)$) to maximize profits after observing the realized demand curve $Q(P(x), x)$. The first-order condition of the seller's problem in each state x characterizes the optimal flexible price $P^*(x)$[4]

$$M^i \left(\partial_P \pi + \partial_Q \pi \cdot \partial_P Q\right) \mu(x) = 0 \qquad \forall x \in X. \tag{3}$$

[4] It is assumed that $\pi_{PP} < 0$ whenever the probability of a state occurring $\mu(x) > 0$ in these analyses.

Consequently, when both prices and quantities are flexible, price variation does not generate financial risk $\pi_P = 0$. This means that the stochastic discount factor M^i drops out. In this flexible benchmark, both the buyer and the seller's stochastic discount factors are irrelevant.

Linearizing Equation (3) around the expected state $x \to \mathbb{E}[x]$,[5] the first-order condition creates the implicit relationship

$$P^*(x) - P^*(\mathbb{E}[x]) \approx -\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}}(x - \mathbb{E}[x]) \tag{4}$$

where $\bar{\pi}_{Px}$ and $\bar{\pi}_{PP}$ are second-order total derivatives evaluated at the approximation point. This equation states that the optimal flexible price responds to demand and cost shocks with elasticity $-\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}}$. For example, in a standard monopoly pricing problem, the firm's ideal flexible price is a desired markup $-\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}}$ over the realized marginal costs of producing the traded good, which is a coordinate of the state variable x.

Sticky Price Solution—The pricing schedule in Equation (1) restricts the price to be invoiced in a set of currencies. In other words, the producer currency price can only change due to exchange rate fluctuations. This creates a "tracking error" between the sticky and flexible price, formally $\text{Var}(P^*(x) - P_0 - \beta s)$, that causes the seller to leak monopoly profits. A seller can retain its pricing power by choosing the invoicing share that best approximates the flexible price. To a second-order approximation, this is a regression of exchange rates onto Equation (4), a result which is often referred to by the literature as the "exchange-rate passthrough" (ERPT) onto the flexible price.[6]

Proposition 1 (Engel 2006)**.** *For flexible quantities $\delta = 0$, to a second-order approximation, the optimal currency of invoicing replicates the exchange rate passthrough (ERPT) onto flexible prices*

$$\beta^*_{\delta=0} \approx -\underbrace{\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} b_{xs}}_{\text{Real Hedging}}, \qquad \text{as } x \to \mathbb{E}[x]$$

where $b_{xs} := \frac{\text{Cov}(x,s)}{\text{Var}(s)}$ is the regression coefficient of exchange rates and the state variable x.

This is a seminal result covered in both of the handbook chapters on currency invoicing (Burstein and Gopinath, 2014; Gopinath and Itskhoki, 2022). By projecting Equation (4) onto the exchange rate, a producer can recover an approximate implementation of their desired markup over marginal cost. The real risks of international trade, such as uncertainty over future costs and demand shocks, affect the desired currency of invoicing.

[5] The details of this perturbation are in Appendix A.1.

[6] An older literature on hedging refers to this as the "minimum-variance hedging formula" (Johnson, 1960; Stein, 1961). Anderson and Danthine (1981) note that the formula solves a second-order approximation of an expected utility framework but does not provide a proof of the result.

The key takeaway is that with flexible quantities $\delta = 0$ and sticky prices, financial markets do not affect the optimal currency invoicing decision. This is because at the approximation point, the discount factors M^i and M^j vanish from the seller's problem. To a second-order approximation, currency choice does not generate financial risk $\bar{\pi}_P = 0$, so incentives for risk sharing do not affect currency choice. In the limiting case of flexible quantities, the currency of invoicing choice only hedges the real risks of international trade. In the next section, I show that this *classic irrelevance result* breaks down when FX hedging markets are imperfect and quantities are sticky.

1.3 Sticky Quantities

This section is the primary result of the paper: when quantities are sticky, the cost of FX hedging affects the optimal currency of invoicing. To develop this insight, I provide a model of costly FX hedging often used the literature on international finance to relate each agent's stochastic discount factors with FX financial markets.

Definition 2. For each SDF M^i and M^j, there exists an **Euler equation wedge $\boldsymbol{\tau}$** such that the producer R and local currency bond R^* Euler equations hold exactly,

$$1 + \tau^i := \mathbb{E}\left[M^i R\right]; \qquad 1 + \tau^j := \mathbb{E}\left[M^j R\right] \tag{Home Bond}$$

$$1 + \tau^{i*} := \mathbb{E}\left[M^i R^* S\right]; \qquad 1 + \tau^{j*} := \mathbb{E}\left[M^j R^* S\right]. \tag{Foreign Bond}$$

Define the seller and buyer's **effective forward rate** F_i and F_j as

$$F_i = \frac{R}{R^*}\frac{1+\tau^{i*}}{1+\tau^i}; \qquad F_j = \frac{R}{R^*}\frac{1+\tau^{j*}}{1+\tau^j}. \tag{Effective Forward Rate}$$

The **relative cost of FX hedging** is the difference $\Delta_{ij}F = F_i - F_j$.

The 4 Euler equations represents each agent's first-order condition from the optimal portfolio allocation problem across the home and local currency bonds. Both SDFs are denominated in the producer currency, so the returns are also written in producer currency units, that is, R for the producer currency bond and R^*S for the local currency bond. Because in the background of the model there is a seller and buyer each making two investment decisions, there are four Euler Equations. When both seller and buyer are able to invest and the capital markets are perfect, the equations should hold exactly, such that $1 = \mathbb{E}[MR] = \mathbb{E}[MR^*S]$ for both SDFs M^i and M^j, i.e. $\boldsymbol{\tau} = \mathbf{0}$.

These four wedges $(\tau^i, \tau^{i*}, \tau^j, \tau^{j*})$ capture capital market imperfections in reduced-form. A wedge affects an agent's perceived rate of return, so a higher wedge is as if the agent faced a higher interest rate. For example, if the Dijon producer borrows in the euro at 4% but Trader Joe's borrows in the euro at 5%, it is as if Trader Joe's faces a 1% euro Euler equation wedge, expressed as $\tau^j = 0.01$. The wedges reflect the interest rate cost of financial frictions and may be caused by interest rate taxes, transaction costs, or corporate finance frictions that shift the perceived rate of return on these securities. In Appendix B, I provide examples of how these frictions appear as wedges. My modeling device of wedges allows me to study the importance of frictions in financial hedging without specifying the microfoundation.

In perfect financial markets, the covered interest parity relationship implies that the forward rate is determined by the ratio of the nominal interest rates in each currency. Specifically, by short-selling a local currency bond and using the proceeds $1/R^*$ to invest in the producer currency bond in the first period, each agent creates an investment strategy that guarantees a financial hedge in the second-period with the payoff $\frac{R}{R^*} - S$. Because entering a forward contract creates a perfectly substitutable payoff of $F - S$, where the forward rate F is also fixed in advance, the forward rate must equal the return on the synthetic hedging strategy $F = \frac{R}{R^*}$ when markets are perfect.

I define an effective forward rate to capture the existence of FX financial market imperfections for each agent. The rate depends on the ratio of the *perceived* nominal interest rates, which incorporates the Euler equation wedges. These effective forward rates represent each agent's valuation of FX risk as implied by their stochastic discount factor. Indeed, one can verify that the effective forward rate can be rewritten as each agent's certainty equivalent of FX risk: defined as the discounted present value of future exchange rate realizations $F_i = \mathbb{E}\left[\frac{M^i}{\mathbb{E}[M^i]} S\right]$ for the seller and $F_j = \mathbb{E}\left[\frac{M^j}{\mathbb{E}[M^j]} S\right]$ for the buyer.

Finally, I define the relative cost of FX hedging $\Delta_{ij}F$ to capture the differential valuations of FX risk between a seller and a buyer. An increase in the relative cost $\Delta_{ij}F$ corresponds to a net increase in the seller's valuation of the exchange rate risk compared to the buyer. In perfect FX markets, the relative cost of FX hedging should be zero. Indeed, if I set the Euler equation wedges to $\boldsymbol{\tau} = \mathbf{0}$, each agent's effective forward rate becomes the forward rate that is implied by covered interest parity relationships ($F_i = F_j = \frac{R}{R^*}$) and the relative cost of FX hedging is zero ($\Delta_{ij}F = 0$). In general, if the seller and the buyer can frictionlessly trade the same forward rates $F_i = F_j = F$, they equalize their valuation of FX risk through trading the forward rate. The differences in their valuation of FX risk reflect a set of possible frictions in these markets, which I discuss in Section 3.2.

I now derive the main result of this paper. It is a formula that relates the desired exchange rate passthrough to the real and financial risks of trade invoicing.

Theorem 1. *Let quantities be sticky $\delta > 0$. To a second-order approximation the optimal* *currency of invoicing replicates the ERPT onto flexible prices*

$$\beta^* \approx -\left(\underbrace{\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} b_{xs}}_{\textit{Real Hedging}} + \underbrace{\frac{\delta \partial_P \bar{\pi}}{\bar{\pi}_{PP}} \frac{\Delta_{ij} F}{\text{Var}\,(s)}}_{\textit{Financial Hedging}}\right), \qquad \textit{as } x \to \mathbb{E}\,[x]\,.$$

Where $\delta \partial_P \bar{\pi} = \bar{\pi}_P$ *is the quantity of financial risk.*

When quantities are sticky $\delta > 0$, the optimal currency choice interacts with financial hedging through the relative cost of FX hedging $\Delta_{ij} F$. The seller posts prices in the currency that minimizes trade costs associated with FX risk.[7] The relative cost of financing hedging $\Delta_{ij} F$ is in percentage points and thus scales with the amount of financial risk $\delta \partial_P \bar{\pi}$. I provide a parameterization of this equation in an Armington trade model in Section 1.4.

To provide intuition, I sketch the proof for Theorem 1 and leave the technical details in Appendix A.1. Begin by considering the solution to the flexible price and sticky quantity problem. When quantities are sticky, prices affect demand through an ex-ante and ex-post demand channel:

$$\frac{dQ\,(x)}{dP\,(x)} = \delta \cdot \underbrace{\frac{dQ_{ea}}{d\mathbb{E}\,[M^j P]} M^j \mu\,(x)}_{\text{Ex Ante Demand}} + (1-\delta) \cdot \underbrace{\frac{dQ_{ep}\,(x)}{dP\,(x)}}_{\text{Ex Post Demand}} \qquad \forall x \in X.$$

This ex-ante demand channel is the product of two effects. The first effect $dQ_{ea}/d\mathbb{E}\,[M^j P] < 0$ captures how an increase in the expected discounted price lowers ex-ante demand. The second effect is the buyer's SDF $M^j > 0$, which quantifies how much an increase in the realized price $P\,(x)$ affects the discounted price. This second effect implies that the buyer lowers their ex-ante demand if trade is expensive in "bad" states, i.e. when their marginal utility of wealth is high.

The first-order condition of the seller's optimal flexible price on discounted profits $\mathbb{E}\,[M^i \pi]$ now internalizes an effect on ex-ante demand

$$M^i \cdot \underbrace{(\partial_P \pi + \partial_Q \pi \cdot \partial_P Q)}_{\text{Financial Risk } \pi_P} + M^j \cdot \underbrace{\mathbb{E}\left[M^i \partial_Q \pi \cdot \partial_{\mathbb{E}[M^j P]} Q\right]}_{\text{Ex Ante Price Elasticity}} = 0 \qquad \forall x \in X. \tag{5}$$

In the first term, the standard monopoly pricing tradeoff appears, as in Equation (3). The new term relates to ex-ante demand $\partial_{\mathbb{E}[M^j P]} Q\,(P, \mathbb{E}\,[M^j P]\,, x) < 0$ and depends on the relative

[7] This endogenizes a price-to-market effect (Atkeson and Burstein, 2008; Burstein, Lein and Vogel, 2023) When a seller transacts with a buyer, the insurance premium they can charge depends on the buyer's ability to absorb the risk of exchange rates. Thus, mark-ups are variable by market and producer size.

SDFs of the seller M^i and buyer M^j.[8] The key property of this equation is that it can be re-arranged to show that the seller faces financial risk from prices $\pi_P > 0$ even at the optimal flexible price. The valuation effect $\partial_P \pi$ is no longer perfectly offset by real risk $\partial_P Q$ because prices internalize an effect on ex-ante demand $\partial_{\mathbb{E}[M^j P]} Q \leq 0$.

Linearizing Equation (5) around the expected state $x \to \mathbb{E}[x]$, the first-order condition creates the implicit relationship

$$P^*(x) - P^*(\mathbb{E}[x]) \approx -\left(\underbrace{\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} (x - \mathbb{E}[x])}_{\text{Markup}\times\text{Marginal Cost}} + \underbrace{\frac{\bar{\pi}_P}{\bar{\pi}_{PP}} \left(\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]} \right)}_{\text{Risk-Sharing Concession/Premium}} \right). \tag{6}$$

The optimal flexible price changes with the seller and buyer's incentives to share financial risk, given by the difference in the normalized SDFs $\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}$ multiplied by the quantity of financial risk $\bar{\pi}_P$. This generalizes Equation (4) where risk-sharing dropped out due to the assumption of flexible quantities $\bar{\pi}_P = 0$. However, this risk-sharing term will also drop out in the case of perfect risk-sharing, where the seller and buyer already agree on the marginal utility of wealth in each state $M^i = M^j$.

This risk-sharing term $\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}$ is the central mechanism of this model. It implies that the seller's optimal flexible price adjusts with the buyer and seller's marginal utility of wealth, as captured by the stochastic discount factors. When risk-sharing is perfect $M^i = M^j$, the term vanishes because both parties agree on the marginal utility of wealth state-by-state. However, if there is uninsurable risk, the seller and buyer may disagree on their stochastic discount factors $M^i \neq M^j$, which creates a demand for insurance transfers. When quantities are fixed, the insurance transfer must happen through prices.

The final step of this proof is applying a linear projection of the linearized flexible price $P^*(x)$ onto exchange rates s. The projection of exchange rates onto the risk-sharing term combines the Euler equation wedges in Definition 2,

$$\text{Cov}\left(\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}, s \right) = F_i - F_j := \Delta_{ij} F. \tag{7}$$

This final step rearranges the covariances in terms of the discounted expectation of exchange rates $\mathbb{E}[Ms]$. Intuitively, the equation states that the covariance of the exchange rate to the buyer and seller's relative SDFs must capture the relative cost of FX hedging. If the seller and buyer could freely trade FX hedging instruments, they would self-insure using the home

[8] The probability $\mu(x)$ integrates out because the flexible price in each state $x \in X$ affects demand across all states.

and local currency bond. To the extent that they cannot perfectly insure themselves with these instruments, their valuations of exchange rate risk will differ $F_i \neq F_j$. The buyer may have a higher valuation of exchange rate risk than the seller $F_j > F_i$, for example. Because the relative cost of FX hedging is negative $\Delta_{ij}F < 0$, the optimal currency of invoicing would be pushed towards the producer currency $\beta = 0$ which allows the buyer to assume the risk of FX fluctuations.

In the special case of perfect FX risk sharing or flexible quantities, a Modigliani-Miller irrelevance result holds. Financial hedging does not affect firm value when quantities are flexible or capital markets are perfect.

Corollary 1. *To a second order, financial hedging is irrelevant only if*

1. *Quantities are flexible* $\delta = 0$*; or*
2. *FX risk sharing is perfect* $\Delta_{ij}F = 0$.

1.4 Application: Armington Model

I illustrate the baseline model with a simple example. In this example, I specialize firm profits to have constant marginal costs to producing the traded good and assume the buyer has a constant elasticity of demand. These assumptions follow Armington (1969) and are standard in small open economy settings with sticky prices.

Details of the Example—The seller produces its output using a constant marginal cost production function. For the purposes of this illustration, the marginal cost curve can be described as $C = C_0 (1 + \gamma \cdot s)$ where C_0 is the first-period producer currency price of inputs, γ is the share of local currency inputs, and s is the local currency exchange rate. When the local currency exchange rate appreciates, marginal costs increase in proportion to the share of foreign inputs. The seller's profit function is increasing in prices and decreasing in marginal costs

$$\pi (P, Q, C) = (P - C) Q.$$

The buyer's value function is given by a constant elasticity of substitution demand curve $V (Q, \mathcal{V}) = \mathcal{V} \cdot Q^{-1/\sigma}$ where $\sigma > 1$ is the demand elasticity and $\mathcal{V} > 0$ is an exogenous willingness-to-pay index, which is an element of the random vector x. In this example, the state $x \in X$ is the 3-dimensional random vector consisting of the seller's marginal cost, the buyer's value index, and the percentage change in the local currency exchange rate $\langle C, \mathcal{V}, s\rangle \in \mathbb{R}^3$.

Solution—In Appendix A.2, I apply Theorem 1 to solve for the second-order approximation of the optimal pricing and currency invoicing decisions. Define $\mathcal{M} = \frac{\sigma}{\sigma-1} > 1$ as the seller's desired markup. The firm's optimal sticky price is the expectation of the firm's optimal flexible price, given by

$$P_0 + \beta\mathbb{E}[s] = \mathcal{M} \cdot C_0 (1 + \gamma\mathbb{E}[s]). \tag{8}$$

In expectation, this is some markup $\mathcal{M}$ over the marginal cost $C(x)$, thus deriving Equation (8). Incentives for risk sharing do not affect the expected flexible price, which in equilibrium has both concessions and premiums that wash out on average.[9] Its desired local currency exchange rate passthrough follows the formula

$$\beta^*/P_0 \approx \underbrace{\gamma}_{\text{Real Hedging}} + \underbrace{\frac{\delta}{1-\delta}(\mathcal{M}-1)\frac{\Delta_{ij}F}{\text{Var}(s)}}_{\text{Financial Hedging}}. \tag{9}$$

Equation 9 internalizes the real and financial risks of trade invoicing. The first term γ captures real hedging. It states that a seller will invoice international goods trade in the currency of its inputs. Indeed, if inputs are only in the producer currency $\gamma = 0$, this force lowers the seller's desired local currency passthrough. This is a form of operational hedging that is sometimes referred to as "marry and netting," in which the FX risk associated with inputs and outputs are matched (Ito et al., 2018). In the opposite extreme, where all costs are in local currency $\gamma = 1$, the producer passes the risk of the local currency exchange rate in costs through the optimal price.

This real hedging effect captures the producer's desire to maintain its profit margin. For example, in a call-off contract where quantities are determined by Trader Joe's on a per-need basis, the Dijon producer prefers to invoice goods in the euro to maintain its desired markup over local French wages (set in advance and denominated in euros). If unit prices are invoiced in dollars, the dollar-euro exchange rate could depreciate against the producer. Trader Joe's may strategically increase call-off order sizes for Dijon mustard precisely when the dollar cost has fallen, forcing the producer to sell at a lower markup. The producer would perfectly hedge this risk if it could set its invoicing decision to track the FX passthrough of costs $\gamma = \beta$.

The financial hedging mechanism manifests itself through the second term in Equation (9), i.e. $\frac{\delta}{1-\delta}(\mathcal{M}-1)\frac{\Delta_{ij}F}{\text{Var}(s)}$. Depending on the relative cost of FX hedging $\Delta_{ij}F$, the seller shifts its invoicing decision to efficiently shares FX risk. This risk sharing effect is in pro-

[9]Notably, it does not depend on the degree of real rigidities δ. Nonetheless, there is an implicit price concession because the buyer's demanded quantity is actually higher with risk-sharing, as captured by the ex-ante demand curve $Q^m_{ea} := \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right)$. This ex-ante demand curve internalizes a lower price when the price P covaries negatively with the stochastic discount factor M^j.

portion to the term $\frac{\delta}{1-\delta}(\mathcal{M}-1)$ which vanishes with zero markups $\mathcal{M}=1$ or fully flexible quantities $\delta=0$. Positive markups $\mathcal{M}>1$ are important because the firm needs to make profits to be exposed to financial risk. Formally, this is captured in the assumption that $\pi(\cdot,0,\cdot)=0$ and $\partial_Q\pi>0$, which implies that the firm makes positive profits in equilibrium. Because the demand elasticity is finite $\sigma>1$, the constructed example satisfies this technical restriction $\mathcal{M}>1$.

As the fraction of fixed quantities δ tends to one, the currency of invoicing decision responds dramatically to the cost of financial hedging $\lim_{\delta\uparrow 1}\frac{\delta}{1-\delta}=+\infty$ because the trade contract can be perfectly hedged. One can construct an "arbitrage" trade that illustrates this point. Consider a firm-fixed price trade contract that promises to deliver 10 units of Dijon mustard at a unit price of $1,000 in 3 months. Payment is made on the delivery of the goods (standard in open-account procedures). If the producer can hedge at a dollar-euro 3 month forward rate of 1:1, they can add a $10,000 notional FX forward contract to swap the dollars into a guaranteed payment of €10,000. The FX forward rate therefore makes the producer indifferent between posting a unit price of $1,000 and €1,000. Thus, invoicing is determined by which unit price, the buyer, Trader Joe's prefers to pay. For example, if Trader Joe's can enter into a higher forward rate on euro-dollar hedges $F_i<F_j$, they can buy goods in the euro and hedge back the contract into the dollar, lowering total trade costs.

In the model, the euro-invoicing "arbitrage" arises because the Dijon producer and Trader Joe's face different costs of FX hedging. For Trader Joe's to assume the risk of euro-dollar exchange rate fluctuations $\Delta_{ij}F<0$, the effective forward rates must satisfy

$$F_i<F_j.$$

This inequality implies that Trader Joe's places a higher value on euro-dollar risk than the Dijon producer, which tilts the trade contract toward euro invoicing. Section 3.2 presents empirical evidence on frictions in FX hedging that generate such differential valuations. Small buyers and sellers, who often depend on a single bank for both hedging and financing, face especially imperfect markets. These frictions are amplified in emerging markets, where currency non-convertibility and capital controls further impede FX risk sharing. In these settings, the effective forward rate reflects equilibrium differences in risk-bearing capacity.

2 Multiple Currencies and Pricing Regions

It is standard for sellers to fix prices in one currency rather than choosing a basket of currencies β^*, and when choosing a currency, to consider "dominant" currencies such as the dollar. In this section, I develop the theoretical conditions under which a firm optimally chooses one currency among multiple currencies. I apply the solution to the discrete choice problem to study the pricing regions implied by the Armington trade model illustrated in Section 1.4. Through the lens of this classic model, I show how producer (PCP), local (LCP), or dominant (DCP) currency pricing arises as a hedge against the real risks of international trade and the financial risks of FX markets.

2.1 Discrete Choice Solution

Denote $\mathcal{C}$ as the set of currencies. There are $|\mathcal{C}| = n+1$ currencies and, therefore, n bilateral pairs with the producer's currency. The exchange rate is an n-dimensional random vector $S \geq 0$, each a coordinate of the state variable $x \in X$. The relative cost of FX hedging is an n-dimensional scalar $\Delta_{ij}F$. Denote the variance-covariance matrix of the exchange rates by the invertible matrix $\Sigma \in \mathbb{R}^{n \times n}$.

In this setting, the invoicing problem chooses a vector of exchange rate passthrough denoted by β. The pricing schedule follows

$$P(s) = P_0 + \beta \cdot s \qquad \beta \in \{\mathbf{0}, P_0 e_1, ..., P_0 e_n\}.$$

In this equation, P_0 is the scalar price level, $\cdot$ is the dot product, e_l is the l-unit vector in $\mathbb{R}^n$, and $s = S - 1$ is the foreign currency appreciation for each of the n bilateral pairs. The numeraire for unit prices is the producer's currency. For example, if prices are fixed in the producer's currency, the seller chooses the vector of zeros $\beta = \mathbf{0}$. If prices are fixed in the dollar, the seller chooses the dollar unit vector $\beta = P_0 e_{\$}$.

The solution to the multiple-currency discrete choice problem is given by

Proposition 2. *Denote β^* as the vector of optimal ERPT in Theorem 1,*

$$\beta^* = -\left(\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} b_{xs} + \frac{\delta \partial_P \bar{\pi}}{\bar{\pi}_{PP}} \Sigma^{-1} \Delta_{ij} F \right)$$

where $b_{xs} = \Sigma^{-1} \text{Cov}(s, x)$. If β^* is finite, then to a second-order approximation, the solution to the discrete choice currency problem equivalently solves

$$\beta \in \arg \min_{\bar{\beta} \in \{\mathbf{0}, P_0 e_1, ..., P_0 e_n\}} \left(\bar{\beta} - \beta^* \right)^{\intercal} \Sigma \left(\bar{\beta} - \beta^* \right) \qquad \textit{as } x \to \mathbb{E}[x].$$

This result states that the optimal currency of invoicing choice minimizes the total exchange rate risk induced by the discrete choice problem. When the seller cannot post prices in a basket of currencies, as implied by the vector β^*, it creates additional exchange rate risk that depends on the full variance-covariance matrix of exchange rates Σ. The result implies that a seller-buyer pair may often post prices in a third currency, such as the dominant currency, if it approximately implements the desired passthrough of the producer and local currency. For example, a Korean producer selling goods to a Chinese importer may have a desired passthrough of 1/2 won and 1/2 renminbi. It is quantitatively possible that choosing the dollar minimizes the total exchange rate risk because the won-dollar exchange rate is positively covaried with the won-renminbi exchange rate.

Proposition 2 generalizes discrete choice solutions in the literature. Consider the special case of the producer versus local currency pricing problem studied in Engel (2006); Gopinath, Itskhoki and Rigobon (2010). When there is only one foreign currency $n = 1$, the result reduces to a 1/2 threshold rule in which the optimal currency of invoicing is PCP if $\beta^*/P_0 < 1/2$ and LCP otherwise. For example, a producer that desires a local currency pricing passthrough of 20% would opt to entirely pricing in its producer currency. My contribution is to show that this formula holds for a more general contracting environment including the baseline model where $1 > \delta \geq 0$.[10] In the next section, I show how the regions for producer, local, and dominant currency pricing depend on the real and financial risks of trade invoicing. These regions model the particular currency choice that a buyer-seller pair would choose based on exogenous characteristics, such as the fixed share of quantities and the share of foreign currency inputs into the production process for the traded good.

2.2 Producer and Local Currency Pricing

In the producer versus local currency pricing problem, the seller invoices trade in either its own currency ($\beta = 0$) or the currency of the destination market ($\beta = P_0$). This section illustrates the quantitative implications of the model for this discrete choice. To make it concrete, I consider a Korean exporter selling to a Chinese importer within the Armington framework of Section 1.4, where profits are

$$\pi = (P - C_0(1 + \gamma s))\, Q$$

and the unit value is $V = \mathcal{V} \cdot Q^{-1/\sigma}$, with elasticity of demand $\sigma > 1$ and exogenous value index $\mathcal{V}$. The producer currency is the won, the local currency is the renminbi, s denotes the bilateral exchange rate, γ the expenditure share on renminbi inputs, and $\Delta_{ij}F$ the relative cost of won-renminbi hedging. States of the world $x \in X$ are indexed by $\langle C, \mathcal{V}, s \rangle$. Rewriting

[10] Note that $\lim_{\delta \to 1} \beta^* = \pm\infty$ for $\Delta_{ij}F \neq 0$ and is otherwise indeterminant. Proposition 2 holds for all δ values except exactly 1. Appendix A.1.5 discusses this case—the solution is bang-bang in the sign of $\Delta_{ij}F$.

the seller's desired renminbi passthrough,

$$\beta^*/P_0 = \gamma + \frac{\delta}{1-\delta}(\mathcal{M}-1)\frac{\Delta_{ij}F}{\text{Var}\,(s)},$$

it depends on both the expenditure share of local-currency inputs and the financial hedging motive. The seller chooses LCP whenever desired passthrough exceeds one-half. I parameterize the long-run trade elasticity as $\sigma = 5$, which implies a profit margin of $\mathcal{M} - 1 = 0.25$ (Head and Mayer, 2014).

Figure 1 shows pricing regions in the classic Armington model. The left panel sets $\Delta_{ij}F = 0$, eliminating the financial hedging motive, while the right panel relaxes this assumption $\Delta_{ij}F \neq 0$.[11] This contrast isolates the role of financial hedging when buyers and sellers value exchange rate risk differently.

Each point on the plot shows the optimal invoicing decision for a given buyer-seller pair. The axes capture heterogeneity across pairs. The y-axis is the won expenditure share: higher shares shift invoicing toward the won (PCP), consistent with real hedging, since pricing in the cost currency preserves markups. The x-axis is the share of fixed quantities: moving right raises the seller's desired passthrough of financial risk, since sticky quantities increase exposure to exchange rate fluctuations.

In the left graph, I plot the pricing regions implied by a buyer-seller pair that can financially hedge their FX risk at the same price. Having access to FX forwards, the buyer-seller pair equalizes their effective forward rate with the market forward rate $F_i = F_j = F$, which implies that the relative cost of FX hedging is zero $\Delta_{ij}F = 0$. For example, this plot illustrates the invoicing decisions of a large Korean producer and Japanese importer, each with access to financial services through multiple banking relationships. In this setting, the buyer-seller pair avoids distorting the invoicing currency choice of the trade contract to share FX risk because they can engage in side bets through the FX forward market. The pricing regions in purple (renminbi) and yellow (won) depend only on the won expenditure share of the Korea producer. Sellers with the same won expenditure share $1 - \gamma$ make the same invoicing decision independent of quantity stickiness δ. Local currency pricing is chosen only when the won producer has a majority of its inputs in the renminbi.

In the right graph, I plot the pricing regions implied by a buyer-seller pair when the buyer has a lower valuation of renminbi exchange rate risk $F_i > F_j$. This inequality implies that the seller is willing to invoice in the local currency and bear the risk of fluctuations in the renminbi exchange rate. For example, this illustrates how a financially sophisticated

[11] The magnitude of $\Delta_{ij}F$ does not affect the qualitative pattern: as $\delta \to 1$, financial hedging alone governs invoicing.

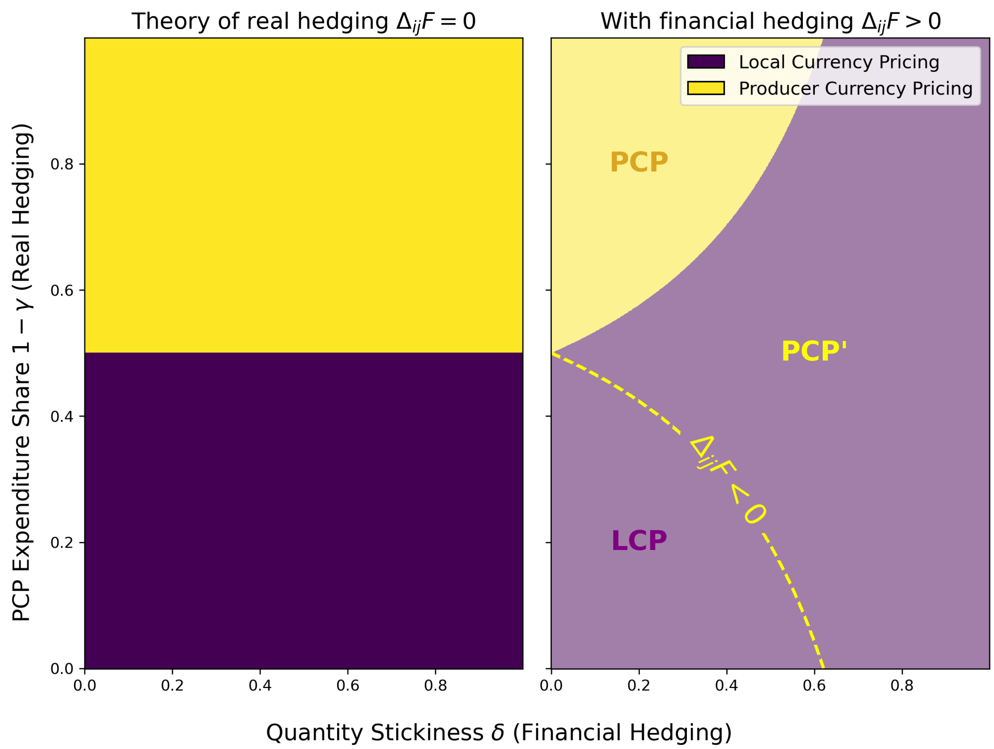

Figure 1: This figure shows the optimal invoicing choice for a particular buyer-seller pair. The right graph shades the **PCP** and **LCP** regions when the seller has a higher valuation of FX risk $F_i > F_j$, and labels the region **PCP'** under the flipped parameterization $\Delta_{ij}F < 0$.

seller and a small buyer, with limited access to FX hedging, would choose their optimal currency of invoicing. In this configuration, the buyer is unwilling to bear the risk of won exchange rate fluctuations, resulting in a shift toward local currency pricing. In this graph, the won invoicing regions are replaced by renminbi invoicing regions, reflecting this demand for insurance. The demand may even cause sellers with only won inputs $\gamma = 0$ to switch to renminbi invoicing, as quantity stickiness increases $\delta \to 1$. As the share of fixed quantities goes to 1, the trade invoicing decision becomes a perfect substitute for FX risk sharing and will only reflect the sign of the relative cost of FX hedging.

The right graph also plots alternative boundaries for pricing regions, when the relative cost of FX hedging is negative $\Delta_{ij}F < 0$. In this alternative configuration, the Chinese importer has a higher valuation of FX risk than the Korean exporter. The reversal of this financial hedging mechanism pushes buyer-seller pairs in the **PCP'** region to switch from **LCP** (purple) to **PCP** (yellow), as outlined by the dotted yellow line. This shifts the optimal currency of invoicing to the Korean won, which allows the importer to assume the risk of exchange rate fluctuations. The pricing regions dramatically change by introducing the financial hedging motive, and moreover, they depend on the financial characteristics of the buyer-seller pair.

In the Armington trade model, the classic theory of real hedging implies LCP whenever the local currency expenditure share exceeds one-half. My theory of trade invoicing depends on both production expenditure shares (real hedging) and the relative valuation of FX risk across buyers and sellers (financial hedging), yielding novel empirical predictions. For example, firm size should correlate with invoicing choices: large exporters can hedge through forwards and invoice in foreign currency, while small importers, who are unable to FX hedge at competitive rates, often prefer LCP to minimize FX risk (Ito et al., 2018; Lyonnet, Martin and Mejean, 2022).[12]

The financial hedging mechanism is especially relevant in emerging markets. Gopinath et al. (2020) show that dollar passthrough is higher in emerging market economies. The Bank of Korea, for instance, reports that 80 percent of Korean exports are invoiced in dollars, a pattern typical for other emerging markets (Son, 2023). This trade dollarization reflects the limited liquidity of emerging market currencies subject to capital controls. For example, China maintains separate on-shore and off-shore markets for the renminbi, which restricts hedging capacity. While such policies increase government control over the exchange rate, they increase the appeal of invoicing in a global currency such as the US dollar. In the next section, I extend the model to allow buyer-seller pairs to choose a third currency, the "dominant" currency, to capture this effect.

2.3 Dominant Currency Pricing

In international goods trade, prices are often invoiced in a third currency such as the US dollar. DCP remains a puzzling feature of trade invoicing. For example, it suggests that in the data, a Korean exporter selling goods to a Chinese importer will often invoice goods in the dollar, despite the dollar exposing both the producer and the importer to the risks of trade invoicing. DCP matters because it implies that the relative price of home and foreign goods is tied to movements in the dollar exchange rate, amplifying the passthrough of US monetary policy into a country's terms of trade, i.e. the relative price of import and export goods, even among countries not directly trading with the United States.

The standard theories of dominant currency pricing are based on the network effects of trade invoicing. Mukhin (2022) demonstrates that the theory of real hedging can rationalize DCP when taking into account the invoicing decisions of competitors. By deriving the unit value function from a Kimball aggregator, he demonstrates that the currency of

[12] I provide suggestive evidence for this mechanism in Appendix C.1. I find that in the cross section of firm survey responses, firms that report FX financial hedging and export goods with sticky quantities tend to invoice in the foreign currency.

invoicing responds to the invoice decisions of competitors of the same destination market, which tends to strengthen dollar invoicing shares. Gopinath and Stein (2021) and Coppola, Krishnamurthy and Xu (2023) demonstrate that the theory of financial hedging can rationalize DCP through the equilibrium cost of financial borrowing. These theories assume fixed quantities and model the desire to match dollar trade invoicing with cheap dollar debt as a synthetic hedge. The decision to export in the dollar increases the liquidity of the dollar, which further reduces the cost of dollar debt and amplifies dollar invoicing.

In this illustration, I show what happens when I combine the real and financial hedging theories into a model of optimal trade invoicing. I adopt the same setup as in Section 2.2: a Korean exporter sells a differentiated good to a Chinese importer with constant elasticity of substitution preferences. I now allow the exporter to choose from a broader set of invoicing currencies: the won as the producer currency, the renminbi as the local currency, and the dollar as the dominant currency. This modifies Equation (9) into a two-dimensional vector of the desired dollar and renminbi FX passthrough:

$$\beta^*/P_0 = \begin{pmatrix} \gamma_\$ \\ 0 \end{pmatrix} + \frac{\delta}{1-\delta}\left(\mathcal{M}-1\right)\Sigma^{-1}\Delta_{ij}F. \tag{10}$$

In Equation (10), a fraction $\gamma_\$$ of the Korean exporter's marginal costs are invoiced in dollars, while none are directly tied to the renminbi exchange rate. Thus, the won and dollar hedge the real risks of cost shocks. In addition, following Proposition 5, the desired passthrough depends on the covariance matrix of exchange rates, Σ. Finally, the pricing region is no longer determined by the simple 1/2-threshold rule but instead follows the discrete-choice solution across multiple currencies characterized in Proposition 2.

Figure 2 plots the pricing regions for the Korean producer. These pricing regions display how a particular seller-buyer pair would choose the currency of trade invoicing. The x-axis and the y-axis vary the won expenditure share and the quantity stickiness of a buyer-seller pair, as in Figure 1.

I start by analyzing the case of perfect FX risk-sharing on the left. This figure represents the optimal invoicing decisions for buyer-seller pairs that can perfectly trade FX forwards. By being able to hedge the exchange rate risk associated with trade invoicing, the seller and buyer avoid distorting the currency of invoicing of the trade contract. Consequently, the buyer-seller pair chooses DCP when the expenditure share on dollar inputs exceeds 1/2; otherwise, it chooses PCP. For example, Korean manufacturers that import dollar-invoiced silicon chips are more likely to invoice goods in the dollar. Their invoicing decision depends on the invoicing decision of the upstream firms, which creates strategic complementarities in invoicing decisions. Real hedging pushes towards dollar invoicing due to this endogenous

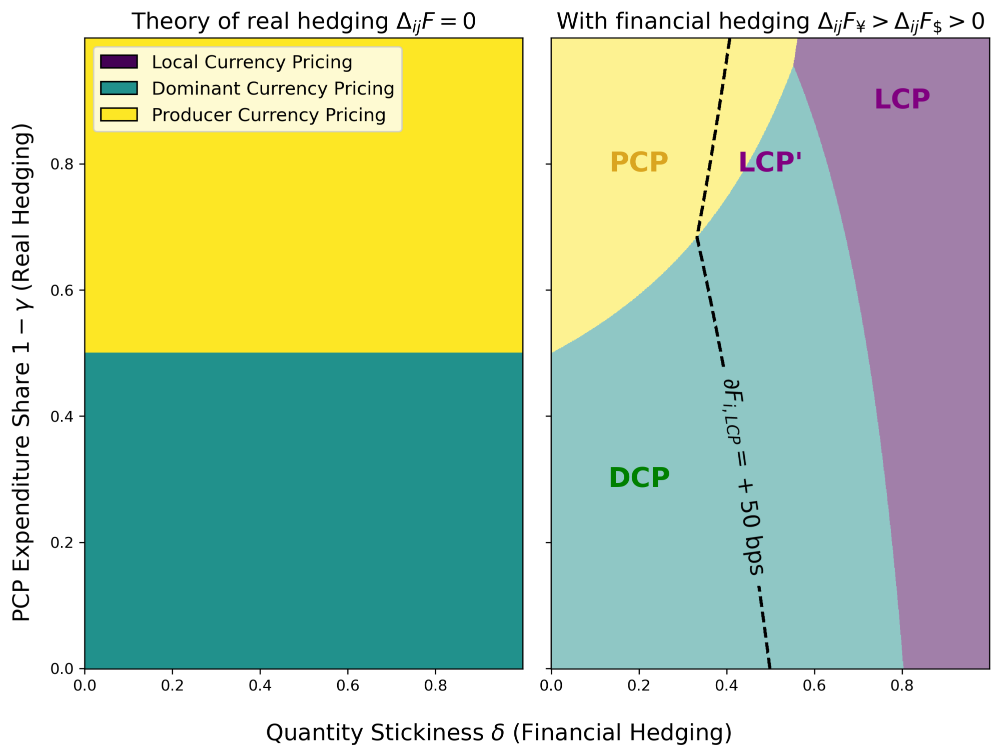

Figure 2: This figure shows the optimal invoicing choice for a particular buyer-seller pair. The right graph shades the **PCP**, **DCP**, and **LCP** regions when the relative cost of FX hedging satisfies $\Delta_{ij}F_{¥} > \Delta_{ij}F_{\$} > 0$, and labels the regions under the alternative parameterization **LCP'** where the local currency forward rate is subsidized by 50bps for the exporter.

network amplification force.

In the right graph, the pricing regions implied by a buyer-seller pair are shaded, where in this configuration, I assume that the relative cost of FX hedging follows the ordering $\Delta_{ij}F_{¥} > \Delta_{i,j}F_{\$} > 0$. This ordering reflects the buyer's preference to first pay in the renminbi, then the dollar, and finally the won.[13] The buyer is willing to pay a premium for the renminbi and a smaller premium for the dollar. This creates regions of renminbi and dollar invoicing that grow in size with quantity stickiness (along the x-axis). Indeed, this pattern is more apparent for Korean exporters with a PCP expenditure share exceeding 1/2, in the top half region of the plot. An increase in quantity stickiness first causes a transition from won invoicing to dollar invoicing, then dollar invoicing to renminbi invoicing. This intermediate transition through the dollar occurs because the dollar jointly hedges real and financial risks. Indeed, in the ideal passthrough Equation (10), the dollar appears in both the real hedging term $\gamma_{\$}$ and the financial hedging term $\Delta_{ij}F_{\$}$.

[13]This parameterization captures an empirically salient feature of the international banking system. Gopinath and Stein (2021) discuss how importers tend to save in deposits in the local and dominant currencies, creating a preference to promise payment in the local and dominant currency for goods trade.

In the right graph, I also include an alternative parameterization where I consider a 50 bps subsidy to the exporter's forward rate. The dotted line represents the new pricing region boundaries that occur with this policy. The **LCP'** region labels the areas that used to be **PCP** and **DCP**. This counterfactual illustrates how easing FX hedging costs can induce exporters to invoice in the renminbi to improve risk sharing with importers. In Section 3.3, I analyze the data and document precisely such a shift from dollar to renminbi invoicing in Korean exports to China following the introduction of the offshore RMB-KRW exchange rate market in Seoul. I estimate how this policy led to an increase in aggregate RMB invoicing shares of roughly 4-6 percent, consistent with the comparative statics of the financial hedging mechanism developed here.

Quantity stickiness and imperfect FX hedging yield novel predictions for the optimal currency of invoicing. The classic theory of real hedging chooses a third currency such as the US dollar only when the seller is downstream in a trade network that heavily relies on dominant currency pricing. It predicts that firms with a majority of their inputs in the won will default to producer currency pricing. Financial hedging changes these invoicing predictions in intuitive ways. Buyer-seller pairs may adopt dominant currency pricing when quantities are partially fixed because the dollar also hedges the financial risks of trade invoicing. It is a liquid currency with desirable FX hedging properties. However, when quantities are close to fully fixed, the optimal invoicing decision will default to renminbi invoicing (LCP) to maximize the gains from FX risk sharing.

3 Empirical Plausibility

This section provides empirical evidence supporting the financial hedging mechanism. I begin by documenting the two key ingredients required for this mechanism, quantity stickiness and imperfect foreign-exchange (FX) hedging, and discuss which types of buyer-seller pairs are most affected by these frictions. I then turn to a case study of Korean exports to China, where the establishment of a bilateral RMB-KRW market led to a shift in invoicing from the dominant currency (dollar) toward the local currency (renminbi). This natural experiment illustrates how changes in the relative cost of FX hedging alters the optimal currency of invoicing in practice, linking FX financial markets to trade invoicing patterns in the data.

3.1 Evidence on Quantity Stickiness

The strength of the financial hedging mechanism is proportional to the degree to which prices generate financial risk. My paper demonstrates that, in order for prices to generate financial risk, quantities must be sticky. I use this section to document evidence on quantity stickiness at the aggregate and contract level.

3.1.1 Aggregate Evidence on Quantity Stickiness

A basic feature of international trade is that quantities have to be fixed before shipment. Bills of lading, which document each shipment, specify the quantity of goods ex-ante, making the transaction binding before delivery. This stands in contrast to spot transactions in domestic retail markets, where prices are posted but quantities are chosen and paid for simultaneously (e.g., at a grocery store). Thus, in international trade, it is difficult to imagine that all quantities are fully flexibly determined at the time of sale.

Gopinath and Rigobon (2008) show that the majority of quantities are fixed in advance in the US data. They document that in the Bureau of Labor Statistics data for market transactions, 40% of exports and 29% of imports specify a quantity range. In contrast, 58% of exports and 70% of imports prespecify quantity (i.e. a fixed number of units or a container). There are no trade contracts that leave quantities fully unspecified. In other words, commitment on both prices and quantities is an empirically salient feature of trade contracts. Interestingly, the share of fixed quantities does not correlate with price stickiness. Together, this evidence suggests a substantial degree of fixed quantities in the data δ, and, moreover, heterogeneity in this share between imports and exports within the United States.

Aggregate quantity stickiness is at the core of some of the seminal insights in international trade and exchange rates. Magee (1973)'s work on J-curves documented how trade balances adjusted to an exchange rate depreciation when prices were set in a foreign currency. Magee argued that trade deficits can often worsen before improving following an exchange rate depreciation because, in "the very short run, quantities are essentially fixed." A depreciation of the producer currency first increases the total home-currency value of foreign-currency denominated imports, before causing the demand for imports to fall more than prices.

Leading estimates of the expenditure-switching effect of exchange rates are consistent with short-run quantity stickiness. Amiti, Itskhoki and Konings (2022) studies price and quantity patterns among Belgium exporters, using census-level data. Their Figure IV shows that the effect falls between 0.446 and 1.709, increasing monotonically with the time horizon of trade. Similar estimates are obtained in other contexts (Berman, Martin and Mayer, 2012; Auer et al., 2019; Auer, Burstein and Lein, 2021; Barbiero, 2021; Auer et al., 2023). This indicates that quantities are stickier in the short run, further suggesting that financial hedging is more important for short-term trade contracts.

3.1.2 Contract-Level Evidence on Quantity Stickiness

Quantity stickiness varies between product varieties and industries, implying a heterogeneity in the importance of financial hedging. In this section, I apply this heterogeneity to understand trade invoicing patterns along three dimensions: differentiated and non-differentiated goods, global value chains, and standalone contracts.

Empirically, differentiated products are more often invoiced in either the producer or local currency, while non-differentiated goods are disproportionately invoiced in the dominant currency (McKinnon, 1979; Kamps, 2006; Goldberg and Tille, 2008). Formally, a differentiated variety is one in which the consumer cannot fully substitute the good with another, in contrast to standardized exports such as commodities. A large literature documents that differentiated goods exhibit quantity "stickiness" in trade data (Broda, Limao and Weinstein, 2008; Soderbery, 2018). The intuition is straightforward: differentiated goods are typically traded in markets where resale or reallocation is limited once production and delivery occur. My theory of financial hedging rationalizes the tendency for differentiated goods to be priced in the producer and local currency. With little scope for ex-post quantity adjustment, both buyers and sellers bear more foreign exchange rate risk when the goods are invoiced in a foreign currency. This pushes the buyer and seller to price in the producer or local currency, as a third currency such as the US dollar would expose both parties to FX financial risk.

Global value chains provide an additional source of quantity stickiness that amplifies the role of financial hedging. de Soyres et al. (2021) and Fernandes and Winters (2021) show that participation in global value chains dampens the expenditure-switching effect of exchange rate movements: firms reliant on imported inputs are unable to immediately expand output in response to exchange-rate-induced shifts in demand. This rigidity heightens the relevance of financial hedging in currency choice. Consistent with this mechanism, Yoshimi et al. (2024) documents that in Japan's automobile industry, intra-firm exports are often invoiced in the buyer's local currency, with the parent firm absorbing exchange rate risk. Similarly, Ito et al. (2018) report survey evidence that Japanese multinationals employ local currency pricing in global value chains to centralize FX risk management and insulate subsidiaries from yen-denominated exposures. Taken together, these findings suggest that as global value chains have become central to international trade, financial hedging motives have grown in importance in determining the currency of invoicing.

The financial hedging mechanism provides an additional lens for interpreting pricing decisions in standalone trade contracts. In the absence of repeated interactions through a framework trade agreement, the standalone contract must jointly negotiate prices and quantities prior to delivery. In contrast, framework trade agreements determine the price schedule ex-ante and allow buyers to make limited call-off orders on an ex-post basis. Consequently, it is unsurprising that local currency pricing is common in large trade transactions, where indivisible goods must be sold on a one-off basis (Goldberg and Tille, 2009, 2016; Junko et al., 2024). As in the global value chain story, exporters writing standalone trade contracts use LCP to absorb FX risk away from importers. Exporters are less likely to price in the producer or dominant currency because in a standalone contract, there are no real risks associated with trade invoicing—by fixing quantities in advance, the seller realizes its costs before delivery.

3.2 Imperfections in FX Financial Hedging

The financial hedging mechanism in my framework operates under the assumption that FX hedging is imperfect: otherwise the buyer and seller share the same valuation of exchange rate risk $\Delta_{ij}F = 0$. This subsection reviews key imperfections documented in the literature and discusses their implications for buyer-seller pairs.

The most liquid currency pairs exhibit persistent deviations from covered interest parity (CIP). Keynes (1923) provides one of the earliest observations of such deviations, and a large modern literature has confirmed their continued presence. In frictionless FX markets, CIP ensures that the forward rate F equals the interest rate differential between the two currencies, R/R^*.[14] Within my model, CIP holds if and only if FX financial hedging is perfect $\boldsymbol{\tau} = 0$. Empirically, however, CIP violations remain substantial even in major currency pairs (Du and Schreger, 2016; Du, Tepper and Verdelhan, 2018). The magnitude and direction are related to the changes in the demand and the supply for FX hedging (Ivashina, Scharfstein and Stein, 2015; Bahaj and Reis, 2022; Moskowitz et al., 2024*a*,*b*; Anderson, Du and Schlusche, 2025; Siriwardane, Sunderam and Wallen, 2025). Although recent advances have improved the measurement of CIP deviations and documented their relationship to banking regulation, long-term evidence suggests that CIP deviations were a persistent fact before the Global Financial Crisis (Obstfeld and Taylor, 2003; Olivier et al., 2025).

Additionally, financial hedging is imperfect due to transaction costs and market segmentation in over-the-counter (OTC) FX markets. Hau et al. (2021) show that bid-ask spreads in EUR/USD forwards decline sharply with a client's financial sophistication: single-banked clients face average transaction cost spreads of roughly 67 basis points, while clients trading through multiple intermediaries face near-zero costs. Similarly, Ranaldo and Somogyi (2021) document pricing asymmetries consistent with informational frictions and OTC market fragmentation. In my framework, such heterogeneity implies that the effective forward rates F_i and F_j available to a buyer and a seller depend on their access to financial intermediaries and

[14] Recall that the first period exchange rate is normalized to 1.

the breadth of their banking relationships. Large exporters with multiple banking partners can obtain more competitive forward rates, which in conjunction with foreign currency trade invoicing, can be used to improve FX risk sharing (Ito et al., 2018; Lyonnet, Martin and Mejean, 2022).

Transaction costs in FX markets tend to create a special vehicle-currency role of the U.S. dollar. Somogyi (2022) find that roughly 13 percent of global dollar trading volume arises from triangular trades between two non-dollar currencies. Krugman (1980) relates this phenomenon in FX markets to the dominance of the dollar, showing that currencies with lower transaction costs endogenously become vehicle currencies in trade. In my model, which embeds a financial hedging channel, this implies that buyer-seller pairs operating in illiquid currency markets optimally invoice in the dollar, leading to dominant currency pricing. This mechanism helps explain the elevated dollar passthrough observed in trade between emerging market economies (Gopinath et al., 2020).

The liquidity of FX markets of emerging market economies is related to capital controls and currency nonconvertibility policies. These policies restrict the ability of private agents to freely transact or hedge in foreign currency (Cerutti, Obstfeld and Zhou, 2021). Historically, such controls took the form of parallel exchange rates that implicitly taxed or subsidized invoicing and settlement decisions (Vries, 1987, 1996). Modern versions include explicit taxes on foreign-currency debt (e.g., Brazil's IOF tax) and quantitative limits on FX transactions (e.g., cash-transfer caps on the Turkish lira). Governments employ these tools to segment capital markets and mitigate the risk of capital flight during financial stress. As capital controls have become an accepted element of macroprudential policy (Basu et al., 2020), they now play a central role in shaping the cost and feasibility of FX hedging that feeds back into the use of the dollar in trade invoicing.

3.3 DCP to LCP: Korean Exports to China

I now turn to a case study of Korean exports to China, where the establishment of a bilateral RMB-KRW market led to a shift in invoicing from the dominant currency (dollar) toward the local currency (renminbi). In July 2014, Chinese president Xi Jinping visited Korea for the Korea-China summit. During this time, the two countries met to strengthen bilateral relationships and improve trade and financial access. In particular, during the summit, China and Korea signed a deal to open an offshore renminbi market in Seoul and an expansion of the RFQII quota, which increased the scope of investments in yuan-denominated assets from Korean investors. China is South Korea's largest market of exports, largest source of imports, and largest destination of overseas investment, while South Korea was China's third-largest trading partner and fifth-largest source of foreign investment in 2013.

The goal of establishing a bilateral "CNK" facility between the renminbi (RMB) and the won (KRW) was to facilitate cross-border payments in the producer and local currency. Exporters receiving Chinese yuan could directly convert the currency into Korean won instead of converting it through the US dollar.[15] Before establishing a bilateral foreign exchange market, a Chinese importer would need to convert their onshore yuan into the US dollar at the government-controlled foreign exchange rate. Then, the Chinese importer would send the dollars to a correspondent bank that would deposit the dollars into the Korean exporter's bank account. The lengthy process exposed the export price to the dollar exchange rate and introduced additional uncertainty in the effective exchange rate between the yuan and the won.

The financial hedging effects of this offshore exchange rate market were widely discussed in the context of currency invoicing. In the EM Macro Daily research note at Goldman Sachs dated March 3, Managing Director Goohoon Kwon wrote:

> We see several supporting cases for the creation of a yuan-won market in Korea. First, a CNK market would facilitate yuan settlement in bilateral trade, which would in turn help boost Korean exports to Chinese domestic markets by removing currency risks for Chinese importers. Yuan settlement would make it easier for Chinese importers to buy goods from Korea, especially small and medium-sized importers for domestic markets (not for re-exports) who tend to face difficulties with hedging their currency exposure.

Kwon's argument emphasizes the risk-sharing nature of yuan invoicing. It is especially relevant for Korean exporters selling to small Chinese importers, who face financial costs in won hedging. Because China maintains capital controls that segment its onshore and offshore yuan markets, a CNK facility affects the *relative* cost of FX hedging by lowering hedging costs exclusively for Korean exporters, as opposed to Chinese importers. Through the lens of the model, this would correspond to an increase in the effective forward rate of the exporter $\partial F_i > 0$, holding the rate constant for the importer $\partial_j F = 0$. Theorem 1 indicates that by financial hedging, this policy increases RMB invoicing shares all else equal.

I study the effect of financial hedging on invoice patterns by interpreting this summit as a change in financial hedging. I define the "treatment" effect of establishing a bilateral exchange rate market as the difference between the RMB invoice shares in Korean exports to China and a "control" group that captures what the shares would have been without such

[15] Korea's deputy finance minister, Jung Eunbo, stated that increasing the use of yuan for trade settlement reduced transaction costs for local exporters.

Control Unit			Control Unit		
Region	**Currency**	**Weight**	**Region**	**Currency**	**Weight**
Europe	EUR	0	Middle East	KRW	0
—	GBP	0	—	USD	0
—	KRW	0	Southeast Asia	HKD	0.114
—	RUB	0.56	—	KRW	0
—	USD	0	—	SGD	0
Japan	JPY	0	—	USD	0
—	KRW	0.295	United States	KRW	0
—	USD	0	—	USD	0
Latin America	BRL	0.031	Other	KRW	0
—	KRW	0	—	USD	0
—	USD	0	**TOTAL**		1

Table 1: This table represents the region-currency (ISO3) weights $wts_{reg,cur}$ assigned by the synthetic control method, used for computing counterfactual invoicing shares. Weights are assigned to match the quarterly preintervention path of invoicing shares, from December 2010 until September 2014. Source: Bank of Korea.

a facility,

$$Treatment_t = \beta^{RMB}_{KOR \to CHN,t} - \mathbb{E}\left[\beta^{RMB}_{KOR \to CHN,t} \mid \text{No CNK Facility}\right].$$

I estimate this counterfactual control using a synthetic control method (Abadie, Diamond and Hainmueller, 2010). In my specification of this synthetic control method, I construct the counterfactual share using Korean export invoicing shares to all other regions excluding China. Denote $wts_{reg,cur}$ as a set of positive weights that sum to 1, for each region $reg \in \mathcal{J}$ and currency $cur \in \mathcal{C}$. The synthetic control method computes a set of time-invariant weights to construct

$$\mathbb{E}\left[\beta^{RMB}_{KOR \to CHN,t} \mid \text{No CNK Facility}\right] = \sum_{reg \in \mathcal{J}} \sum_{cur \in \mathcal{C}} wts_{reg,cur} \beta^{cur}_{KOR \to reg,t}.$$

My analysis uses aggregate invoicing shares data from the Bank of Korea, where time t is indexed at the monthly frequency. I restrict the set of currencies to include only the producer, local, and dominant currencies for each region. In total, there are 21 control units. Following standard applications of the synthetic control method, I estimate the weights by targeting the preintervention path of the outcome variable, using quarterly invoicing shares from December 2010 until September 2014.

Table 1 shows the weights of each currency-region in the synthetic version of Korean RMB export shares to China. The synthetic control is a weighted average of the Russian ruble,

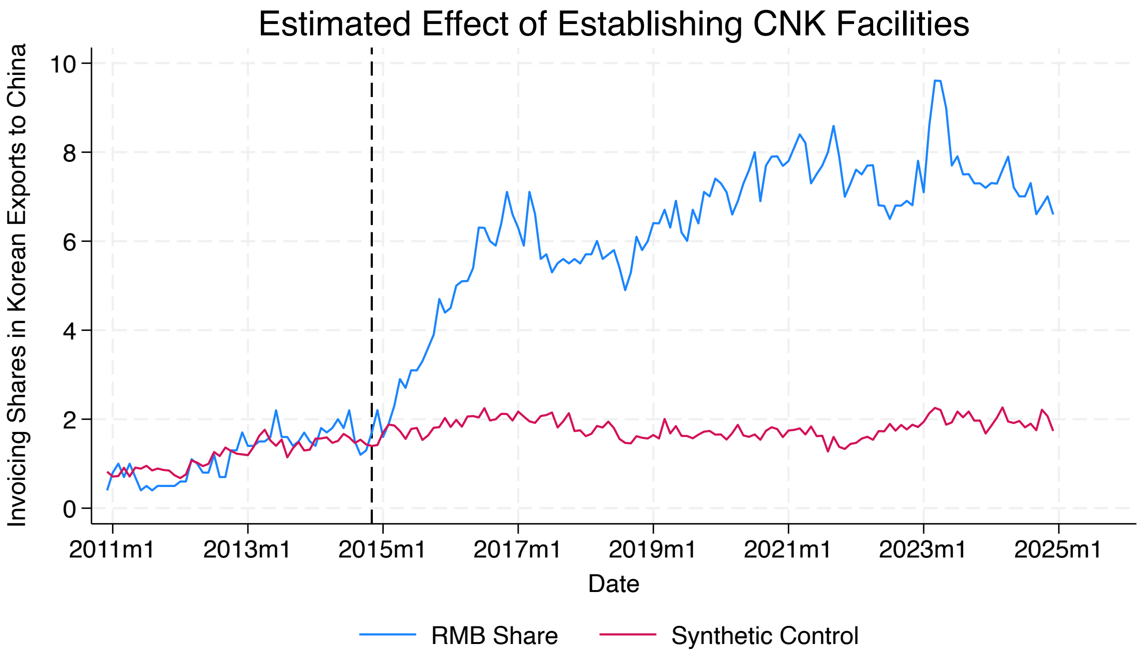

Figure 3: This plots the RMB invoicing shares (in percentage points) in Korean exports to China against a synthetic control group. The synthetic control group is estimated using a weighted average of aggregate Korean export invoice shares, reported in Table 1. The dashed line marks the month in which the CNK facility was established. Source: Bank of Korea.

the Hong Kong dollar, the Brazilian real, and the Korean won invoicing shares to Japan. All other control units obtain zero weights due to a regularization procedure that penalizes positive coefficients. The synthetic weights appear sensible. For example, they suggest that the RMB shares to China are unlike the dollar invoicing shares. Instead, the synthetic control weights put the highest value on exports to Europe in the ruble (i.e. Russia), which is a local currency with moderate liquidity in the 2010s.

Figure 3 plots the RMB invoice shares in the data and the synthetic control. The estimated treatment effect is the difference between the blue and red line. After the CNK market was established, the realized RMB share is 4-6% higher than the synthetic control group, which is flat over time. In other words, by introducing FX liquidity, China was able to push Korea away from a DCP paradigm (dollar invoicing) towards an LCP paradigm (renminbi invoicing). In absence of reform, it was unlikely that the invoicing share would have changed dramatically. From April 2015 to February 2020, the estimated effect is significant at the 5% level based on standard permutation tests reported in Appendix C.3.[16]

[16]In the synthetic control method, significance is computed by constructing a distribution of "random" treatment effects. The procedure samples each of the control units and estimates the random assignment distribution by assigning these control units as if they were the treated observation.

The CNK policy experiment closely mirrors the mechanism in Section 2.3, where a Korean exporter chooses whether to invoice in won, renminbi, or dollars when selling to a Chinese importer. In the model, a decline in the relative cost of renminbi hedging expands the region where local currency pricing (LCP) is optimal, denoted **LCP'** in Figure 2. The synthetic control evidence in Figure 3 provides an aggregate analogue: following the CNK market's introduction, the share of renminbi-denominated Korean exports rises by roughly 4-6 percentage points relative to the synthetic counterfactual. This shift reflects the same underlying mechanism: a reduction in the relative cost of renminbi hedging for the seller that induces more buyer-seller pairs to adopt LCP.

At the aggregate level, the observed increase in RMB invoicing masks considerable heterogeneity across exporters. Firms facing stickier quantities or higher exposure to financial hedging margins are most responsive to changes in hedging costs, while others with flexible quantities remain unaffected. In this sense, the modest but persistent rise in RMB invoicing shares is consistent with a reallocation across heterogeneous buyer-seller pairs rather than a wholesale shift in pricing behavior.

Threats to Causal Inference—To interpret the synthetic control method as a causal estimate, three conditions must be met: no anticipation, convex hull, and no interference. No anticipation is the condition that the treated shares are unaffected before the treatment date. For example, there would be anticipation in shares if the CNK facility was created in response to increasing trends in RMB invoicing and bilateral trade. However, the bilateral CNK market was spurred by de-dollarization efforts, triggered by sanctions against Russia after the Crimean invasion in March 2014.[17] The sanctions against Russia raised concerns that the United States would leverage the US dollar to reduce financial and monetary autonomy as part of its geopolitical strategy. The no anticipation condition is met if the timing of the Crimean invasion was unanticipated among exporters and importers.

Convex hull is the condition that the synthetic control can be represented by a weighted average of the control group's invoicing shares. The preintervention period is matched and places the majority of its weight on LCP shares to other regions.

No interference requires that the control units are unaffected by the treatment. PCP, LCP, and DCP invoicing shares to regions outside China need to be unaffected by the CNK market. For example, it is possible that raising RMB liquidity increases the seller's demand for dollar invoicing in goods trade with Latin America. My theory of financial hedging highlights that this consideration only matters to the extent that it affects the *relative* cost

[17] On October 2014, Chinese Premier Li Keqiang signed 38 agreements on a visit to Moscow in 2014, establishing a three-year currency swap deal worth 150 billion yuan (about $24.5 billion). Since then, the Chinese central bank has steadily decreased its holdings of US treasuries in an effort to de-dollarize.

of FX hedging. This policy reform is a special economic environment to test financial hedging because China maintains capital controls on domestic importers through the onshore-offshore exchange rate markets. The CNK market affects the effective forward rate of the exporter, keeping the importer's valuation fixed ($\partial F_i > 0, \partial F_j = 0$). My interpretation assumes that even if the liquidity of RMB affects the cost of hedging other currencies, it does not change the *relative* cost between the buyer-seller pairs, for other region-currency pairs.

4 Macroprudential Policy: Dollarization Dilemma

I embed my theory of the optimal currency of invoicing into a canonical small open-economy New Keynesian framework with capital controls. This will allow me to explore the interaction between endogenous trade invoicing and optimal macroprudential policy. In this economy, prices are sticky and FX financial hedging is imperfect because the government can tax domestic and foreign-currency capital borrowing and lending.

A classic result in this class of models is that the government optimally taxes foreign currency borrowing, because the foreign currency tends to appreciate in value precisely when the economy enters a recession (Bianchi, 2011; Korinek, 2018; Farhi and Werning, 2014, 2016; Bianchi and Lorenzoni, 2022). In a model where the foreign currency is the dollar, I show that this commonly recommended policy amplifies dominant currency pricing when invoicing is endogenous. In my model, local agents facing a tax on dollar borrowing substitute dollar financial debt into dollar trade invoicing, which acts as a synthetic liability when quantities are sticky. This introduces a "dollarization dilemma" in which the planner must choose between discouraging financial dollarization and trade dollarization.

I demonstrate this dilemma by studying the Ramsey planning problem. The planner's problem describes the government's optimal policies as a function of the equilibrium conditions of the economy. I start by characterizing these conditions to show the patterns of substitution that induce the dollarization dilemma. I then describe the planner's problem in terms of these conditions, and finally derive the optimal capital controls. I contrast these capital controls with the literature by studying the flexible exchange rate example in Farhi and Werning (2016) with a deviation that allows trade invoicing to be endogenously determined.[18]

[18] I extend my insights to a Galí and Monacelli (2005) model with multiple currencies in Appendix D.

4.1 Environment

Households—There are two periods $t = 0, 1$. At $t = 1$, the state of the world x realizes with probability $\mu(x)$. I leave the dependence on x as implicit whenever possible, and instead subscript with time t. For simplicity, I assume there are two currencies: a home currency and the dollar, which represents an international currency used for financial transactions and trade invoicing. The dollar exchange rate is denoted by S_t and normalized to 1 in the initial period $S_0 = 1$. An increase in the exchange rate corresponds to an appreciation of the dollar relative to the home currency.

There is a representative domestic agent with preferences

$$\mathbb{E}\left[\sum_{t=0}^{1} \rho^t U\left(C_{NT,t}, C_{T,t}, N_t\right)\right]$$

where $C_{NT,t}$ denotes the consumption of non-tradables, $C_{T,t}$ denotes the consumption of foreign tradable goods, and N_t denotes the total labor supply. I assume that preferences over consumption are homothetic and the disutility of labor is separable.

Households are subject to the following budget constraint:

$$\begin{aligned} &P_{NT,t}C_{NT,t} + P_{T,t}C_{T,t} + S_t B_t^F + B_t^H \\ &\quad \leq W_t N_t + P_{X,t}X_t + \Pi_t + T_t + \frac{1}{R_t^*\left(1+\tau_t^{BF}\right)} S_t B_{t+1}^F + \frac{1}{R_t} B_{t+1}^H \end{aligned}$$

which states that the total expenditure on non-tradable and tradable goods is less than or equal to wage income $W_t N_t$, profits from domestic non-tradables Π_t, lump sum transfers from the government T_t, commodity endowments X_t exported at an exogenous spot market price $P_{X,t}$, and net borrowing in the home-currency and dollar bonds, B_t^H and B_t^F respectively.[19] Bonds are one-period investments and yield the domestic and dollar risk-free interest rates R_t and R_t^*. In this equation, the price of the non-tradable and the foreign tradable good is given by $P_{NT,t}$ and $P_{T,t}$. The capital control tax on dollar debt is τ_t^{BF}. An increase in τ_t^{BF} increases the local agent's perceived after-tax yield on dollar borrowing.

I also introduce a capital control tax τ_t^{BH} on foreign investments into local-currency debt. There are risk-neutral foreigners who can invest in domestic securities. These foreigners ensure that the uncovered interest rate parity equation must hold despite the presence of dollar exchange rate risk. I suppress the time subscript on both bond taxes because there

[19] To simplify expressions, I assume that commodities are sold on a world market and used to produce the tradable good. Consequently, they are not directly consumed by the local agents of the economy.

are only two periods. Consequently, the uncovered interest parity condition is given by

$$R_0 \mathbb{E}\left[\frac{1}{S_1}\right] = \left(1+\tau^{B^H}\right) R_0^*. \tag{11}$$

The household's first-order conditions are

$$1 = \mathbb{E}\left[\rho \frac{\partial_{C_T} U_1}{\partial_{C_T} U_0} \frac{P_{T,0}}{P_{T,1}} R_0^* \left(1+\tau^{B^F}\right) S_1\right] \tag{12}$$

$$1 = \mathbb{E}\left[\rho \frac{\partial_{C_T} U_1}{\partial_{C_T} U_0} \frac{P_{T,0}}{P_{T,1}} R_0\right] \tag{13}$$

$$\frac{\partial_{C_T} U_t}{\partial_{C_{NT}} U_t} = \frac{P_{T,t}}{P_{NT,t}} \tag{14}$$

$$\frac{W_t}{P_{NT,t}} = -\frac{\partial_N U_t}{\partial_{C_{NT}} U_t}. \tag{15}$$

These four equations represent the equilibrium conditions of the household block.

Firms— I now discuss how the price of non-tradable and tradable goods is set. Tradable and non-tradable goods are produced by a market of domestically-owned competitive firms that combine a continuum of varieties using a constant returns to scale CES technology with elasticity of substitution σ. Varieties are subject to sticky prices and quantities, i.e., prices are set one period in advance and a fraction δ is fixed exogenously.

Each non-tradable variety is produced from labor by a domestically owned monopolist using a linear technology with stochastic productivity A_t. Labor is hired from a competitive market with wage W_t, but each firm pays $W_t(1+\tau_L)$ net of tax on labor. The equilibrium pricing conditions are symmetric across varieties, so I suppress the dependence on the variety,

$$P_{NT,1} = (1+\tau_L)\,\mathcal{M} \cdot \mathbb{E}\left[W_1/A_1\right]. \tag{16}$$

The price of non-tradable varieties reflects a markup $\mathcal{M} = \frac{\sigma}{\sigma-1}$ over the expected wage and productivity. Profits from the non-tradable firms are given by

$$\Pi_t = \left[P_{NT,t} - (1+\tau_L)\frac{W_t}{A_t}\right] C_{NT,t}.$$

Each tradable variety is exported to the local market by a foreign-owned monopolist, using an identical production function with constant dollar marginal costs $C > 0$. These monopolists set prices in a basket of the home currency and the dollar, reflecting each monopolist's desired dollar passthrough. I apply the Armington trade model example in Section 1.4 for each tradable variety to derive the equilibrium pricing condition for the tradable goods

$$P_{T,1} = \mathcal{M} \cdot C\mathbb{E}\left[S_1\right] + \beta\left(S_1 - \mathbb{E}\left[S_1\right]\right). \tag{17}$$

In this equation, β is the dollar FX passthrough onto tradable goods prices and $C\mathbb{E}\left[S_1\right]$ is the expected marginal cost in units of the home currency. The associated dollar passthrough conditions are symmetric across varieties and are given by

$$\beta/\mathbb{E}\left[P_{T,1}\right] = 1 + \frac{\delta}{1-\delta}\frac{\mathcal{M}-1}{\text{Var}\ (S_1)}\frac{R_0}{R_0^*}\left(\frac{1}{1+\tau^{B^H}} - \frac{1}{1+\tau^{B^F}}\right). \tag{18}$$

This equation combines several features of the macroeconomic environment. The first term is a constant (1) to reflect the cost-currency alignment. This characteristic reflects the tendency for real hedging to favor DCP, as argued in Mukhin (2022). The second term captures the financial hedging motive. It increases in the fixed share of the market δ, the profit margin of the exporter $\mathcal{M}-1$, and the relative cost of financial hedging $\Delta_{ij}F$. I have rewritten the relative cost of FX financial hedging in terms of the Euler equation wedges of the seller and buyer. Capital controls appear as Euler equation wedges because they shift the perceived rate of return on home-currency and dollar borrowing for the local and foreign agents. This formalizes the endogenous link between FX hedging costs and trade invoicing.

The **dollarization dilemma** is captured in the equilibrium invoicing condition in Equation (18). Capital controls that fight financial dollarization tend to increase trade dollarization. For example, taxes on dollar borrowing $\tau^{B^F} > 0$ increase the foreign exporter's desired dollar passthrough β, as it induces local agents to use the trade contract as a synthetic dollar liability. Similarly, a subsidy on home currency borrowing $\tau^{B^H} < 0$ increases the desired dollar passthrough β, because it reduces the local agent's willingness to owe an additional unit of risk denominated in the home currency. The strength of this regulatory arbitrage depends on the substitutability of the trade contract and financial debt. When quantities are sticky $\delta > 0$, exporters with pricing power $\mathcal{M} > 1$ can use invoicing to reallocate dollar risk across borders. In contrast, when quantities are flexible $\delta = 0$, this substitution pattern disappears and recovers the standard assumption that prices are exogenously set in the dollar.

Government—The government sets the tax on labor τ_L, capital controls on dollar debt τ^{B^F} and home-currency debt τ^{B^H}, the home currency nominal interest rate R_t, and in addition, it levies lump-sum taxes T_t in period t to balance its budget

$$T_t = \tau_L W_t N_t + \frac{\tau^{B^F}}{1+\tau^{B^F}}\frac{S_t}{R_t^*}B_t^F + \frac{\tau^{B^H}}{1+\tau^{B^H}}\frac{1}{R_t}B_t^H.$$

Equilibrium—An equilibrium takes as given the share of fixed quantities δ, spot price of commodities $P_{X,t}$, endowments of commodities X_t, and the foreign nominal interest rate R_t^*. It specifies consumption of traded and non-traded goods $\{C_{T,t}, C_{NT,t}\}$, labor supply $\{N_t\}$, home currency and dollar debt $\left\{B^H, B^F\right\}$, the price of non-traded goods $\{P_{NT,t}\}$, the price of traded goods $\{P_{T,t}\}$, wages $\{W_t\}$, the labor taxes τ_L, capital controls $\left\{\tau^{B^H}, \tau^{B^F}\right\}$ such that households and firms maximize, the government's budget constraint is satisfied, and markets clear:

$$C_{NT,t} = A_t N_t. \tag{19}$$

These conditions imply that the market for traded goods clears.

4.2 Planner's Problem

I characterize the Ramsey problem of choosing policies that maximize the utility of local agents. The dual approach, which directly chooses policies to influence equilibrium outcomes, is represented by

$$\max_{\left\{\tau_L, \tau_t^{B^F}, \tau_t^{B^H}, R_t, T_t\right\}} \mathbb{E}\left[\sum_{t=0}^{1} \rho^t U\left(C_{NT,t}, C_{T,t}, N_t\right)\right] \qquad \text{(Full Ramsey Problem)}$$

subject to the equilibrium conditions in Equations (11)–(19). This representation combines the government's objective and available instruments with the implementability conditions, which describe the equilibrium allocations as implicit functions of the policies.

To highlight the key economic forces of the model and derive optimal capital controls, I follow Farhi and Werning (2016) by rewriting the Ramsey problem in terms of the primal approach. The primal approach represents the planner's problem as if it were directly choosing allocations, with implicit policies that support these allocations. Indeed, given quantities, Equations (11), (12), (13), (15), and (16) can be used to back out certain prices, wages, and taxes. In addition, these variables do not affect welfare, so they can be eliminated from the planning problem. Using the homotheticity of consumption and the separability of labor disutility, I substitute Equations (14) and (19) and write directly $C_{NT,t} = \alpha\left(p_t\right) C_{T,t}$ and $N_t = C_{NT,t}/A_t$ throughout, where α is some increasing function of the relative price of tradable goods $p_t = P_{T,t}/P_{NT,t}$. Finally, I write the trade invoicing condition in Equation (18) in its primal representation

$$\beta/\mathbb{E}\left[P_{T,1}\right] = 1 - \frac{\delta}{1-\delta} \frac{\mathcal{M}-1}{\text{Var}\left(S_1\right)} \text{Cov}\left(\frac{M_1}{\mathbb{E}\left[M_1\right]}, S_1\right). \tag{20}$$

This representation expresses the relative cost of hedging in terms of the nominal SDF of the local and risk-neutral foreign agents, using Equation (7). The nominal SDF of the local agent is defined by the Euler equations of the household block $M_1 = \rho \frac{\partial_{C_T} U_1}{\partial_{C_T} U_0} \frac{P_{T,0}}{P_{T,1}}$.

The simplified planning problem is given by

$$\max_{\{C_{T,t}, P_{NT,1}, S_1, B_1^H, B_1^F\}} \mathbb{E}\left[\sum_{t=0}^{1} \rho^t U\left(\alpha_t C_{T,t}, C_{T,t}, \frac{\alpha_t}{A_t} C_{T,t}\right)\right] \qquad \text{(Simplified Ramsey Problem)}$$

subject to a standard country resource constraint, which states that the net present value of trade surpluses is equal to zero,

$$P_{T,0} C_{T,0} \leq \frac{1}{R_0^*} B_1^F + \frac{\mathbb{E}\left[1/S_1\right]}{R_0^*} B_1^H + P_{X,0} X_0 \tag{21}$$

$$P_{T,1} C_{T,1} + B_1^F + B_1^H \leq P_{X,1} X_1, \tag{22}$$

and a novel set of implementability conditions (17) and (20) for the price of tradable goods. Importantly, these novel conditions capture the dollarization dilemma. This regulatory arbitrage prevents the planner from separately controlling financing and trade invoicing decisions, creating an additional constraint on implementable allocations.[20]

I define two wedges to guide the analysis of the planner's problem. The first is a standard labor market wedge τ_t^L that captures whether labor is underutilized relative to flexible price allocations. When this wedge is positive, the economy is in a "recession." The wedge is given by the equation

$$\tau_t^L = 1 + \frac{1}{A_t} \frac{\partial_N U_t}{\partial_{C_{NT}} U_t}. \tag{23}$$

When prices are flexible, the labor market wedge is zero in each state because the price of nontradables reflects the ex-post marginal rate of transformation of labor. However, when the price of the non-tradable is too high relative to labor productivity because prices are set in advance, the economy underutilizes labor and there is a non-tradable goods output gap $\tau_t^L > 0$. Macroprudential policy often targets the output gap through monetary policy and labor subsidies.

The second is a novel trade invoicing wedge τ^S introduced by the endogenous link between FX hedging costs and trade invoicing. This scalar represents the *ex-ante* value of increasing

[20] The tradable goods conditions cannot be substituted out because the planner does not have an instrument that separates invoicing from financial markets. This is motivated by existing regulations on trade invoicing. For example, in Argentina, regulations on dollar trade invoicing are implemented using parallel exchange rates, which segment FX financial markets (Ilzetzki, Reinhart and Rogoff, 2019).

the dollar FX passthrough β, which corresponds to a *marginal* transition from local to dominant currency pricing (LCP/DCP):

$$\tau^S = \text{Cov}\left(S_1, \partial_{C_T} U_1 \left[\underbrace{\partial_p \alpha_1 C_{T,1} \tau_1^L}_{\text{Substitution Effect}} - \underbrace{\left(1 + \frac{\alpha_1}{p_1}\tau_1^L\right) C_{T,1}}_{\text{Income Effect}}\right]\right). \tag{24}$$

The trade invoicing wedge in Equation (24) consists of the covariance of the exchange rate with a substitution and income effect. Both effects equal zero when the prices of tradable and nontradable goods are flexibly set.

The substitution effect captures the classic expenditure-switching role of exchange rates. Higher dollar passthrough allows exchange rate movements to shift relative demand between domestic and foreign tradables. The social value of this mechanism is proportional to the labor market wedge, τ_1^L. When dollar appreciations are correlated with recessions, $\text{Cov}\left(S_1, \tau_1^L\right) > 0$, greater passthrough improves welfare: it suppresses demand for imports precisely when labor is underutilized (through an appreciation in the price of foreign tradables), thereby reallocating expenditure toward domestic nontradables. In this case, DCP helps mitigate labor market slack by letting the exchange rate adjust relative prices in a socially desirable direction, echoing the classic argument for flexible exchange rates as a tool to offset nominal rigidities (Friedman, 1953; Mundell, 1961).

The income effect reflects how invoicing alters the net present value of trade surpluses. When prices of tradable goods are sticky in dollars, a stronger dollar raises import costs. The welfare impact of this channel depends on the marginal utility of tradable consumption, $\partial_{C_T} U_1$, and the labor wedge, $\frac{\alpha_1}{p_1}\tau_1^L$. These terms capture the planner's incentive to redirect demand toward recessionary states. The income effect of DCP can be destabilizing, echoing the classic debate over whether currency devaluations are contractionary (Nurkse, 1944, 1945; Krugman and Taylor, 1978). When the dollar appreciates in bad times, such as during flight-to-safety episodes, the higher cost of foreign tradables depresses real tradable income. This is especially harmful for countries that rely heavily on imports during devaluations $\text{Cov}\left(S_1, C_{T,1}\right) > 0$. In such configurations of the model, DCP exacerbates downturns by amplifying the contraction in tradable consumption.

The sign of the trade-invoicing wedge in Equation (24) is ambiguous, reflecting the complex trade-offs of DCP. In many configurations, the net effect of DCP over LCP trades off a stabilizing expenditure-switching effect with a destabilizing income effect. On the one hand, dollar passthrough can alleviate nominal rigidities because flexible exchange rates offer an additional mechanism to clear trade balances. On the other hand, it can destabilize trade balances by causing the price of tradable goods to appreciate against local agents in recessionary periods. I leave the sign of this wedge unspecified to show how the planner's desired policy reflects a preference for (or against) DCP.

4.3 Optimal Capital Controls

I derive the normative predictions for optimal capital control policy in my model with endogenous trade invoicing. The solution closely follows Farhi and Werning (2016), except that it features a trade-off induced by the dollarization dilemma.

Proposition 3. *The optimal tax on dollar borrowing τ^{B^F} and subsidy on home-currency foreign lending τ^{B^H} are characterized by expectations and covariance terms that depend on the labor market wedge τ_t^L and the trade invoicing wedge τ^S,*

$$1+\tau^{B^F} = \mathbb{E}\left[\frac{1+\frac{\alpha_1}{p_1}\tau_1^L+\frac{d\beta/dC_{T,1}}{\mu_1\partial_{C_T}U_1}\tau^S}{1+\frac{\alpha_0}{p_0}\tau_0^L}\right] + \frac{\mathrm{Cov}\left(M_1 S_1, \frac{1+\frac{\alpha_1}{p_1}\tau_1^L+\frac{d\beta/dC_{T,1}}{\mu_1\partial_{C_T}U_1}\tau^S}{1+\frac{\alpha_0}{p_0}\tau_0^L}\right)}{\mathbb{E}\left[M_1 S_1\right]}$$

$$1+\tau^{B^H} = \mathbb{E}\left[\frac{1+\frac{\alpha_1}{p_1}\tau_1^L+\frac{d\beta/dC_{T,1}}{\mu_1\partial_{C_T}U_1}\tau^S}{1+\frac{\alpha_0}{p_0}\tau_0^L}\right] + \frac{\mathrm{Cov}\left(M_1, \frac{1+\frac{\alpha_1}{p_1}\tau_1^L+\frac{d\beta/dC_{T,1}}{\mu_1\partial_{C_T}U_1}\tau^S}{1+\frac{\alpha_0}{p_0}\tau_0^L}\right)}{\mathbb{E}\left[M_1 S_1\right]}.$$

In these equations, M_1 is the local agent's nominal SDF, μ_1 is the probability of the state occurring, and β is the desired dollar passthrough in tradable goods prices. The term $\frac{d\beta/dC_{T,1}}{\mu_1\partial_{C_T}U_1}$ is zero when quantities are fully flexible $\delta=0$, markups are zero $\mathcal{M}=1$, or local agents are risk neutral $M_1=1/R_0$.

This shows that capital controls on home-currency and dollar debt should be different in general. Indeed, the first (expectation) terms on the right-hand side of these formulas coincide. The covariances differ due to the dependency on the dollar exchange rate. The terms in the optimal capital control depend on the two central forces of the model: the labor market wedge τ^L and the invoicing wedge τ^S.

The labor market wedge captures the classic effect of demand externalities. The first (expectation) term states that debt should be discouraged if the economy is more depressed at date 1 than at date 0. The second (covariance) term pushes towards a higher capital control tax on dollar debt than on home-currency debt if the dollar S_1 tends to appreciate when there is a labor market recession $\tau_1^L > 0$. Higher dollar taxes fight financial dollarization, which reduce capital outflows in recessionary periods—reflecting the benchmark demand externality result (Farhi and Werning, 2014, 2016).

There is a new term related to the effect of capital controls on dollar passthrough β that depends on the trade invoicing wedge τ^S defined in Equation (24). This new term is expressed in terms of the marginal effect of shifting tradable consumption between states on the optimal choice of currency invoicing $d\beta/dC_{T,1}$. Capital controls discourage debt financing and cause the reallocation of tradable consumption across states $C_{T,t}$. This changes the local agent's marginal utility of wealth, which simultaneously affects the endogenous desire to share FX risk through the trade contract. The trade invoicing wedge appears in both the first (expectation) and second (covariance) terms of the optimal capital control. The expectation term is common across taxes and captures the possibility that debt (in general) affects the local agent's risk-bearing capacity. The covariance term captures how home-currency and dollar debt differentially affect trade invoicing.

By specifically taxing dollar debt, the planner pushes local agents to save into the states where the dollar appreciates. Take the usual case where this policy results in a lower marginal utility of wealth in states where the dollar appreciates, i.e.

$$\mathrm{Cov}\left(M_1 S_1, \frac{d\beta/dC_{T,1}}{\mu_1 \partial_{C_T} U_1}\right) > \mathrm{Cov}\left(M_1, \frac{d\beta/dC_{T,1}}{\mu_1 \partial_{C_T} U_1}\right)$$

holds in equilibrium.[21] Under this technical assumption, the covariance term states that a higher tax on dollar borrowing τ^{B^F} shifts the economy toward dominant currency pricing. Local agents substitute their demand for dollar financial liabilities with dollar-denominated trade, which acts as a synthetic liability. This substitution occurs because the tax on dollar debt causes local agents to save in the dollar in excess to what they would have done in the laissez-faire equilibrium, one without capital controls, and simultaneously offset the exposure by importing goods in the dollar. This creates a dollarization dilemma in which dollar taxes deter dollar debt but increase dollar passthrough.

The dollarization dilemma has an ambiguous effect on desired capital controls because it depends on whether the planner favors dominant currency pricing, as captured by the sign of the invoicing wedge τ^S. For example, in emerging market economies, policy makers discuss the negative consequences of dollarization in both financing and trade. By fighting dollar financing with capital controls, these countries inadvertently promote dominant currency pricing, which has the destabilizing effect of increasing a country's import prices in bad times. This dilemma is reflected in Proposition 3. Optimal capital controls are dampened relative to the benchmark result when the planner prefers local currency pricing $\tau^S < 0$.

[21] In the proof of this proposition, I demonstrate that this condition typically holds by showing that $d\beta/dC_{T,1}$ is the product of the coefficient of relative risk aversion and the dollar exchange rate. Indeed, the inequality always holds in the dual representation of the planner's problem; in the primal approach, it becomes harder to sign the inequality.

The dollarization dilemma straddles governments in choosing between discouraging either dollar trade or dollar financing. However, the dilemma is less severe in economies where non-tradable prices are flexible. In this setting, the labor supply is employed at the level implied by the marginal productivity of labor ($\tau_t^L = 0$) and the demand-externality terms in Proposition 3 vanish. Capital controls are set to adjust the economy to the socially optimal dollar passthrough β, which purely depends on the income effect of the trade-invoicing wedge τ^S. The optimal capital control will not tilt all the way to this desired dollar passthrough to avoid excessive distortions in risk sharing. When both tradable and non-tradable goods prices are flexibly set, both the labor market and the invoicing wedge is zero $\tau_t^L = \tau^S = 0$, so the laissez-faire economy is optimal $\tau^{B^F} = \tau^{B^H} = 0$.

My model of optimal macroprudential policy when trade invoicing is endogenous introduces a dilemma in capital controls. Though capital controls can often be used to target excessive dollar borrowing, it can also lead to a substitution away from financial securities toward dominant currency pricing. Because the trade contract is an asset, FX interventions affect both the financing and the trade side of the economy. However, the choice of currency of invoicing uniquely affects current accounts and redirects demand between domestic and foreign goods. Future work will need to be done to extend the conclusions of this model to a setting with multiple countries and currencies, thus building an additional avenue to study the implications of producer currency pricing.

5 Contracting Microfoundation

This section microfounds the baseline model in Section 1 and formalizes the optimal currency of invoicing for renegotiation-proof trade contracts. A renegotiation-proof trade contract is a modeling device for studying limited commitment, which is a salient friction in international goods trade (Antràs and Foley, 2015). The particular setting satisfies three purposes. The first purpose is to microfound flexible quantities with limited commitment. In the spirit of Hart and Moore (1988), I demonstrate that flexible quantities cause financial hedging to become irrelevant precisely because risk sharing is non-enforceable.

The second purpose is to illustrate the broader economic mechanism of financial hedging. Relative to the baseline model, I show that financial hedging mechanism does not depend on the assumption of exogenously fixed quantities, but rather on the ability of the parties to commit, which in equilibrium makes optimal quantities sticky across some states of the world. Commitment pushes the agent with a higher risk-bearing capacity to assume the risk of FX fluctuations. This endogenizes the quantity stickiness of the optimal contract. The third purpose is to study a bilateral trade contract between a single seller and a single buyer, which relaxes the uniform pricing assumption.

5.1 Setting

Fix the measure space $(\Omega, \mathcal{F}^x, \mu)$ where $\mathcal{F}^x$ is the sigma algebra generated by the m-dimensional state variable $x : \Omega \mapsto X \subseteq \mathbb{R}^m$. Denote $C(\Omega)$ as the set of real, bounded, and continuous functions with domain Ω and let $x \in C(\Omega)$. Trade is specified by a contract between the seller i and a single buyer j. A contract $(P, Q) \in C(X)^2$ is a price-quantity tuple, where $X := x(\Omega)$ is the co-domain and taken as given.

I assume the same multiple currency setup as in Section 2.1. $\mathcal{C}$ is the set of currencies which has the cardinality $|\mathcal{C}| = n + 1$ and thus there are n-bilateral pairs with the producer currency. The exchange rate is an n-dimensional vector $S : \Omega \mapsto \mathbb{R}^n_{++}$, each a coordinate of x. Consequently, the nominal rigidity restriction is

$$P(s) = P_0 + \beta \cdot s \qquad \beta \in \{\mathbf{0}, P_0 e_1, ..., P_0 e_n\}$$

where P_0 is a scalar producer currency price level, $\cdot$ is the dot product, e_l is the l-unit vector in $\mathbb{R}^n$, and s is the realized foreign currency appreciation for the n bilateral pairs. These unit vectors have the same interpretation as in Section 2.1 in that they restrict the optimal currency of invoicing to be in a particular currency.

The payoffs generated by the trade contract are the profit function for the seller $\pi(P, Q, x)$ and the value of trade for the buyer $v(P, Q, x)$. I assume these functions are analytic in their arguments. In this more general model, I focus on the case where there is exactly one buyer rather than a continuum. One can then aggregate each contract to a market of buyers, so this generalizes the baseline model and introduces pricing-to-market effects. The key difference of this contracting framework is that it must be renegotiation proof. Renegotiation proofness is the condition that the contract is optimally never renegotiated at the time of sale, otherwise inefficient trade is implemented.

Definition 3. Let $\mathbb{D}^{rng} : X \rightrightarrows \mathbb{R}^2$ be the correspondence of implementable deviations. The contract (P, Q) is said to be **renegotiation-proof** if $\forall x \in X$

$$v(P(s), Q(x), x) \geq \max_{\mathbb{D}^{rng}_j(x;P,Q)} v\left(\hat{P}, \hat{Q}, x\right)$$

where $\mathbb{D}^{rng}_j : X \rightrightarrows \mathbb{D}^{rng}$ is the set of deviations that weakly improve seller profits

$$\mathbb{D}^{rng}_j(x; P, Q) := \left\{\left(\hat{P}, \hat{Q}\right) \in \mathbb{D}^{rng}(x) : \pi\left(\hat{P}, \hat{Q}, x\right) \geq \pi(P(s), Q(x), x)\right\}$$

For the rest of this analysis, I take the implementable deviations correspondence $\mathbb{D}^{rng}$ as continuous, exogenous, and inclusive of $(P(s), Q(x)) \in \mathbb{D}^{rng}(x)$. The correspondence would be exogenous if it specified a legal enforcement problem that causes the buyer to default strategically. In such a case, the deviations that the buyer could implement involve exiting the contract when the terms of trade imply a negative value relative to the outside option, which is normalized to 0.

The seller maximizes the risk-adjusted value of the contract.[22]

Definition 4. Given the measure space $(\Omega, \mathcal{F}^x, \mu)$ and implementable buyer deviations correspondence $\mathbb{D}_j^{rng} : X \rightrightarrows \mathbb{D}^{rng}$, the **optimal renegotiation-proof contract** $(P, Q) \in C(X)^2$ is a price-quantity pair that satisfies:

1. Profit maximization

$$\max_{(P,Q)\in C(X)^2} \mathbb{E}\left[M^i \pi(P, Q, x)\right]$$

2. Individual rationality

$$\mathbb{E}\left[M^j v(P, Q, x)\right] \geq 0$$

3. Incentive compatibility $\forall x \in X$

$$v(P(s), Q(x), x) \geq \max_{\mathbb{D}_j^{rng}(x;P,Q)} v\left(\hat{P}, \hat{Q}, x\right)$$

4. Nominal rigidity $\forall x \in X$

$$P(s) = P_0 + \beta \cdot s \qquad \text{s.t. } \beta \in \{\mathbf{0}, P_0 e_1, ..., P_0 e_n\}.$$

5.2 Microfounding Quantities through Commitment

In this section, I show how the renegotiation proof contract is a microfoundation for the analysis in Section 1.2. Flexible quantities occur when the buyer lacks commitment. I now provide a formal definition for sticky quantities in the context of contracts.

Definition 5. Quantities are said to be **flexible** if $\forall x \in X$, there exists an implicit relationship

$$Q^* = Q(P^*(s), x).$$

Otherwise, they are **sticky**.[23]

[22]Note that the participation constraint for the seller never binds since they exercise all bargaining power and profits satisfy limited liability.

[23]This definition of equality is in the almost surely sense.

This definition of flexible quantities subsumes the one provided in Section 1.2. In a trade contract, quantities are flexible when the seller faces a downward sloping demand curve that only depends on the realized price $P(s)$ and state variable x. In the baseline model, quantities were sticky when a fraction of trade was fixed, because demand also depended on the expected price $\mathbb{E}[M^j P]$. I use this definition to derive the analogue to the baseline model parameter δ, which represents the fraction of buyers who fix quantities in advance.

Proposition 4. *If an optimal contract exists, quantities are flexible iff the buyer lacks commitment (renegotiation occurs a.s.)*

$$\mu\left(X^{cmt} := \left\{x \in X : v\left(P\left(s\right), Q\left(x\right), x\right) > \max_{\mathbb{D}_j^{rng}(x)} v\left(\hat{P}, \hat{Q}, x\right)\right\}\right) = 0. \tag{25}$$

Contract renegotiation is the microfoundation for sticky quantities. Recall that in Subsection 1.2, quantities were flexible if and only if $\delta = 0$. In this model, quantities are flexible if and only if the buyer lacks commitment $\mu\left(X^{cmt}\right) = 0$, that is to say, renegotiation occurs almost surely.[24]

Practically, buyers may lack commitment due to legal enforcement frictions. When the buyer is unable to commit, the only feasible contract is one where the seller delivers whatever is demanded at the spot price. Concretely, examine the incentive constraint. Because prices are fixed by the nominal rigidity, it follows that one can reduce the set of deviations to just quantity variation $\mathbb{D}_{jQ}^{rng}\left(x\right) := \left\{\left(\hat{P}, \hat{Q}\right) \in \mathbb{D}_j^{rng}\left(x\right) : \hat{P} = P\left(s\right)\right\}$. This means that in the states where the buyer lacks commitment, the contract optimally implements

$$v\left(P, Q, x\right) = \max_{\hat{Q} \in \mathbb{D}_{jQ}^{rng}} v\left(P, \hat{Q}, x\right) \qquad \forall x \notin X^{cmt}.$$

And by value matching, the implicit function theorem allows Q^* to be rewritten as $Q\left(P\left(s\right), x\right)$ satisfying Definition 5.

However, if the contracting environment features commitment, the seller will optimally share risk. The individual rationality constraint becomes binding implying that

$$\mathbb{E}\left[M^j v\left(P, Q, x\right)\right] = 0.$$

This causes Q^* to be an implicit *functional* of P when it is not constrained by renegotiation. The quantities $Q^* = Q\left[P, x\right]$ then depend on the full set of price realizations.

[24]Evidence from Auer et al. (2023) suggests that the expenditure switching effect increases for lower income households. One explanation for this is that lower income buyers must renegotiate quantities because they are more sensitive to illiquidity.

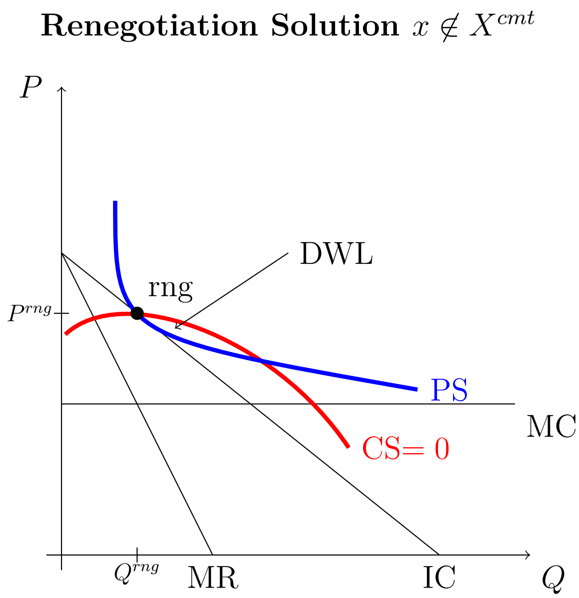

Figure 4: These graphs represent the solution to the contract across states. PS and CS represent the seller and buyer's ex-ante indifference curves. MR and MC are the marginal revenue and cost curves. IC and IC' represents the quantity curve implied by the renegotiation constraint, which is relaxed in the right graph to allow for commitment.

Figure 4 visualizes a price-theory analysis of an optimal contract with renegotiated and committed states. There are two axes, the realized price (y axis) and the quantity (x axis). The primitive is the IC curve, which is a downward sloping renegotiation constraint. Renegotiation is binding on the left graph. The seller takes the renegotiation constraint as given, traces out its implied marginal revenue curve MR, and sets it equal to the marginal cost curve MC. The resulting allocation (P^{rng}, Q^{rng}) maximizes seller profits and satisfies the participation constraint $CS = 0$, but creates a classic monopoly deadweight loss in the oval allocations. The seller is unable to move to these points due to the threat of renegotiation. Instead, it must choose a point on IC.

On the right graph, the buyer and seller are able to commit to the contract terms. IC shifts upwards to IC', locating above the implemented price-quantity pair. Consequently, the seller finds the tangency point between the individual rationality constraint $CS = 0$ and their producer surplus curve PS'. This point is efficient as the seller is now able to knock-back some of the buyer's reduced surplus through generous trade terms in other states. This form of risk sharing improves allocative efficiency because it acts as a state-contingent tariff—rebating the buyer in certain states while extracting surplus when money is more valuable to the seller $M^i > M^j$. Commitment enables a monopolist to jointly specify prices and quantities, which, as argued by Leontief (1946), recovers efficiency in contracts.

Armed with the contracting definition of sticky quantities, I generalize the results of Theorem 1. The proposition determines the optimal currency denomination of a renegotiation proof trade contract when prices are set in a basket of currencies $\beta \in \mathbb{R}^n$.

Proposition 5. *Let $x \to \mathbb{E}[x]$. Define the relative cost of FX hedging conditional on commitment as*

$$\Delta_{ij|X^{cmt}}\mathcal{T} := \mathbb{E}\left[\left(\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}\right) s \mid X^{cmt}\right] \in \mathbb{R}^n.$$

To a second-order approximation, the optimal exchange rate passthrough vector satisfies,

$$\beta^* \approx -\left(\underbrace{\frac{\bar{\pi}_{Px} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{Px}}{\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}} b_{xs}}_{\text{Real Hedging}} + \underbrace{\frac{\bar{\pi}_P \cdot \mu\left(X^{cmt}\right)}{\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}} \Sigma^{-1}\Delta_{ij|X^{cmt}}\mathcal{T}}_{\text{Financial Hedging}}\right)$$

where $\Sigma \in \mathbb{R}^{n\times n}$ is the variance-covariance matrix of exchange rates, $\bar{\pi}_P = \bar{v}_P \frac{\partial_Q \bar{\pi}}{\partial_Q \bar{v}}$ is the quantity of financial risk, and $\bar{v}_P = \partial_P \bar{v} + \partial_Q \bar{v} \partial_P \bar{Q}$ is the financial risk to the buyer. The optimal currency of invoicing is then

$$\beta \in \arg \min_{\bar{\beta} \in \{\mathbf{0}, P_0 e_1, \dots, P_0 e_n\}} \left(\bar{\beta} - \beta^*\right)^{\mathsf{T}} \Sigma \left(\bar{\beta} - \beta^*\right) \qquad \text{as } x \to \mathbb{E}[x].$$

This is a generalized version of Theorem 1 that also applies the discrete choice solution in Proposition 2. Relative to the fixed quantity model, both real and financial hedging depend on the properties of the value function of the buyer v. In the baseline model, the buyer value vanishes because the sticky quantities are fully fixed or flexible, so that $\bar{v}_{PP} = \bar{v}_{Px} = 0$. The seller does not need quantities to be fixed to efficiently share financial risk through currency invoicing. A more general formula will account for how quantities and the buyer value v change across prices and states.

In addition, this formula shows a microfoundation for risk sharing. Quantities are sticky because the buyer can commit to the contract. This causes the seller to expose themselves to financial risk $\bar{\pi}_P > 0$ so that the currency denomination can extract risk sharing gains. The size of these gains are proportional to the measure of committed states $\mu\left(X^{cmt}\right)$ times the relative cost of hedging conditional on commitment $\Sigma^{-1}\Delta_{ij|X^{cmt}}\mathcal{T}$. In the special case where the committed states X^{cmt} are independent of financial conditions, and each state features commitment, the risk sharing gains condition down to $\Sigma^{-1}\Delta_{ij}F$ as before. The optimal currency of invoicing more generally depends on the share of states with commitment and vanishes when quantities are flexible.

6 Conclusion

This paper develops a theory of currency choice in international goods trade with imperfect foreign exchange markets. The paper begins by reexamining the canonical theory of currency choice, which separates real and financial hedging. I show that since this theory assumes flexible quantity contracts, the financial hedging incentive drops out, and foreign exchange markets become irrelevant. If instead quantities are sticky and capital markets are segmented, currency choice reflects financial hedging incentives. My joint theory of real and financial hedging provides a framework for interpreting dominant currency pricing, existing trade invoicing patterns, and is crucial to engineering sound financial and monetary policy.

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Online Appendix for
"Financial Hedging and Optimal Currency of Invoicing"

Oliver Xie

October, 2025

A Proofs and Further Details on the Theory

A.1 Details on Theorem 1 and Proposition 2

To deliver the results on optimal currency invoicing, we formalize the problem as a functional and define the perturbation. It is shown that the classic minimum-variance hedging formula represents the second-order solution of contracting problems. It simplifies proofs and allows us to handle the general contracting problem in Section 5 with ease.

A.1.1 Definitions

By visual inspection, the reduced-form and general contracting problem are special cases of the following convex program.

Definition 6. Fix $(\Omega, \mathcal{F}^x, \mu)$ as the measure space. Denote $C(\Omega)$ as the set of bounded-continuous real-valued functions with the domain Ω endowed with the metric defined by the sup norm. Let $x \in C(\Omega)$ and $a, b \in C(X)$ where $X := x(\Omega)$.

Define the functional $S : C(X)^2 \times C(\Omega) \mapsto \mathbb{R}$ with f being infinitely differentiable (in the Frechet sense) so that S is defined as

$$S[a, b, x] := \int_\Omega f(a, b, x)\, d\mu$$

subject to a set of infinitely differentiable (in the Frechet sense) constraints

$$0 \in G[a, b, x]$$

The convex program is a set of functions $(a^*, b^*) \in C(X)^2$ which maximizes $S[a, b, x]$ subject to satisfying the constraints.

To avoid redundancy, throughout we assume that the differential operator δ is applied in the Frechet sense. It is known that this generalizes the derivative as defined over the real vector space. Moreover, the following fact will be useful.

Fact 1. *The vector spaces $C(X)$ and $C(\Omega)$ with the sup norm metric are Banach spaces.*

This fact is often used in functional analysis. In addition, I require the following set of technical conditions to hold, ensuring the existence of a KKT representation of the infinite-dimensional convex program and well-defined higher-order derivatives.

Lemma 1. *Suppose $b^*[a,x]$ satisfies the Robinson constraint qualification*

$$0 \in int\left\{G[a,b,x] + \delta_b G[a,b,x](b-b^*) : b \in C(X) \times C(\Omega)\right\}$$

so that the set of Lagrange multipliers is non-empty. If the constraint is of the form $0 \in \int_\Omega g(a,b,x)\,d\mu = G[a,b,x]$ for some infinitely differentiable function g, then it follows that there is some neighborhood $\bar{\mathcal{A}}, \bar{\mathcal{B}} \subset C(X)$ and $\bar{X} \subset C(\Omega)$such that $b^*[a,x]$ is of class C^∞ and characterized by some equivalent function $b^*(a,x,\lambda^*[a,x])$. Moreover, the objective can be rewritten as

$$S_b[a,x] = \int_\Omega \tilde{f}(a,x,\lambda^*[a,x])\,d\mu$$

and is also of class C^∞ over the domain $C(X) \times C(\Omega) \mapsto \mathbb{R}$.

Proof. Given the Robinson constraint qualification holds for G, the convex program admits a nonempty, convex, and bounded set of Lagrange multipliers (Theorem 3.9 Bonnans and Shapiro 2000). The Lagrangian multipliers can be represented as such,

$$\mathcal{L}[a,x] := \min_{\lambda \geq 0} \max_{b \in C(X)} \int_\Omega \left[f(a,b,x) - \lambda g(a,b,x)\right] d\mu.$$

For any local maximum, the following first-order conditions are necessary and sufficient

$$\partial_b f(a,b^*,x) - \lambda^* g(a,b^*,x) = 0$$
$$\int_\Omega g(a,b^*,x)\,d\mu = 0.$$

The Implicit Function Theorem (for Banach spaces) as applied to these equations implies that the optimal b can be rewritten as $b^*(a,x,\lambda^*)$ and $\lambda^* = \lambda^*[a,x]$, and moreover for some neighborhoods $\bar{\mathcal{A}} \subset C(X)$ and $\bar{X} \subset C(\Omega)$, the composite $b^*[a,x]$ is of class C^∞. In this neighborhood, we may rewrite

$$S_b[a,x] = \int_\Omega f(a, b^*(a,x,\lambda^*[a,x]), x)\,d\mu$$

which is also of class C^∞ over the domain $C(X) \times C(\Omega) \mapsto \mathbb{R}$ due to the chain rule. By substitution, it follows that there exists some function $\tilde{f}(a,x,\lambda^*[a,x]) = f(a,b^*(a,x,\lambda^*[a,x]),x)$.

□

A.1.2 Minimum Variance Hedging Formula (Johnson, 1960; Stein, 1961)

Lemma 2. *Assume a local optimum exists in some open set* $a^* \in \bar{\mathcal{A}} \subset C(X)$ *and the constraint satisfies the representation in Lemma 1. If the function* $a \in \bar{\mathcal{A}}$ *is a multilinear that takes the form* $a(x) = a_0 + a_{x_l} \cdot x_l$ *where* $a_{x_l} \in \mathbb{R}^A$ *and* x_l *is a subset of the coordinates of* $x \in X$*, the objective achieves*

$$S_b[a,x] - S_b[a^*,x] = \frac{1}{2} A[\mathbb{E}[x]] \cdot \mathbb{E}\left[\left\{a_0 + a_{x_l}^\intercal x_l - \left(a^*(\mathbb{E}[x]) + \partial_x a^*(\mathbb{E}[x])^\intercal (x - \mathbb{E}[x])\right)\right\}^2\right]$$
$$o\left(\|x - \mathbb{E}[x]\|^2\right), \qquad \text{as } x \to \mathbb{E}[x].$$

where $A[x]$ *is defined in the proof. Moreover, when the second-order condition holds* $A[\mathbb{E}[x]] \leq 0$*, the maximizing* $a_{x_l} \in \mathbb{R}^A$ *is given by*

$$a_{x_l} = \text{Var}\,(x_l)^{-1}\,\text{Cov}\,(x_l, x)\,\partial_x a^*(\mathbb{E}[x]).$$

Proof. The fundamental theorem of calculus applies to S_b since $C(X)$ is a Banach space:

$$\begin{aligned} S_b[a,x] &= S_b[a^*,x] + \int_0^1 \frac{dS_b[a^* + (a-a^*)t, x]}{dt} dt \\ &= S_b[a^*,x] + \int_0^1 \frac{\delta S_b}{\delta(a^* + (a-a^*)t)}[a,x]\,\delta(a(x) - a^*(x))\,dt \end{aligned}$$

where the second line follows by way of explicitly expressing the differential. Using $\delta S_b[a^*, x; h] = 0$ from the fact that a^* is a local maximum, we get

$$\begin{aligned} \frac{\delta S_b}{\delta(a^* + (a-a^*)t)}(x) &= \frac{\delta S_b}{\delta a^*}(x) + \int_0^t \frac{\delta^2 S_b}{\delta(a^* + (a-a^*)\tau)^2}(x)\,\delta(a(x) - a^*(x))\,d\tau \\ &= \int_0^t \frac{\delta^2 S_b}{\delta(a^* + (a-a^*)\tau)^2}(x)\,\delta(a(x) - a^*(x))\,d\tau. \end{aligned}$$

Note that because a^* is a local maximum, the variation $\delta^2 S_b$ must be bounded. Combining

$$S_b[a,x] - S_b[a^*,x] = \int_0^1 \int_0^t \frac{\delta^2 S_b}{\delta(a^* + (a-a^*)\tau)^2}(x)\,\delta^2(a(x) - a^*(x))\,d\tau dt.$$

Conjecture that the optimal solution of a satisfies $a(\mathbb{E}[x]) = a^*(\mathbb{E}[x])$.

Let $x \to \mathbb{E}[x]$. It is known that the Taylor Approximation theorem holds for infinitely differentiable functions on Banach Spaces (Theorem 5.6.1 Cartan and Maestro 2017). Define the linearized difference between the solution a^* and the multilinear control a as

$$\Delta a\left(x\right)=a_0+a_{x_l}\cdot x_l-a^*\left(\mathbb{E}\left[x\right]\right)-\partial_x a^*\left(\mathbb{E}\left[x\right]\right)\cdot\left(x-\mathbb{E}\left[x\right]\right)$$

then the second-order approximation is given by

$$S_b\left[a,x\right]-S_b\left[a^*,x\right]=\frac{1}{2}\left(\lim_{x\to\mathbb{E}[x]}\frac{\delta^2 S_b}{\left(\delta a\right)^2}\left(x\right)\right)\Delta a\left(x\right)^2+o\left(\|x-\mathbb{E}\left[x\right]\|^2\right)$$

where $\|\cdot\|$ is the sup norm over the space of continuous functions.

Denote the function

$$\hat{a}\left(x'\right)=\begin{cases}a^*\left(x\left(\omega\right)\right) & \left\{\omega\in\Omega:x\left(\omega\right)\neq x'\left(\omega\right)\right\}\\ a\left(x\left(\omega\right)\right) & \Omega\setminus\left\{\omega\in\Omega:x\left(\omega\right)\neq x'\left(\omega\right)\right\}\end{cases}.$$

Rewrite the total second-order derivative in terms of the pointwise second-order derivatives, which are defined by differentiating with $\hat{a}$. Integrating across states, this becomes

$$\left(\lim_{x\to\mathbb{E}[x]}\frac{\delta^2 S_b}{\left(\delta a\right)^2}\left[a,x\right]\right)\Delta a\left(x\right)^2=\int_\Omega\int_\Omega\frac{\delta^2 S_b}{\delta\hat{a}\left(x'\right)\delta\hat{a}\left(x''\right)}\Delta\hat{a}\left(x'\right)\Delta\hat{a}\left(x''\right)$$

where each pointwise derivative is given by

$$\begin{aligned}\frac{\delta^2 S_b}{\delta\hat{a}\left(x'\right)\delta\hat{a}\left(x''\right)}\left[a,x\right]&=\partial_{aa}\tilde{f}d\mu 1_{x'=x''}+2\partial_{\lambda a(x')}\tilde{f}\partial_{a(x'')}\lambda d\mu\\&\quad+\left(\int\partial_\lambda\tilde{f}d\mu\right)\partial_{a(x')a(x'')}\lambda+\left(\int\partial_{\lambda\lambda}fd\mu\right)\partial_{a(x')}\lambda\partial_{a(x'')}\lambda.\end{aligned}$$

By totally differentiating the constraint $\int g\left(a,b^*\left(a,x,\lambda^*\left[a,x\right]\right),x\right)d\mu$ it follows that

$$\partial_{a(x')}\lambda^*=-\frac{\partial_a g\left(x'\right)+\partial_b g\left(x'\right)\partial_a b^*\left(x'\right)}{\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu}d\mu$$

and moreover

$$\begin{aligned}
\partial_{a(x')a(x'')}\lambda^* = &-\frac{\partial_{aa}g\left(x'\right)1_{x'=x''}+\partial_{ab}g\left(x'\right)\left(\partial_a b^*\left(x'\right)1_{x'=x''}+\partial_\lambda b^*\left(x'\right)\partial_{a(x'')}\lambda\right)}{\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu}d\mu \\
&-\frac{\partial_{ba}g\left(x'\right)\partial_a b^*\left(x'\right)1_{x'=x''}+\partial_{bb}g\left(x'\right)\partial_a b^*\left(x'\right)\left(\partial_a b^*\left(x'\right)1_{x'=x''}+\partial_\lambda b^*\left(x'\right)\partial_{a(x'')}\lambda\right)}{\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu}d\mu \\
&-\frac{\partial_b g\left(x'\right)\left(\partial_{aa}b^*\left(x'\right)1_{x'=x''}+\partial_{a\lambda}b^*\left(x'\right)\partial_{a(x'')}\lambda\right)}{\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu}d\mu \\
&+\frac{\partial_a g\left(x'\right)+\partial_b g\left(x'\right)\partial_a b^*\left(x'\right)}{\left[\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu\right]^2}d\mu\left[\partial_{ba}g\left(x\right)\partial_\lambda b^*\left(x\right)+\partial_b g\left(x\right)\partial_{\lambda a}b^*\left(x\right)\right]d\mu. \\
&+\frac{\partial_a g\left(x'\right)+\partial_b g\left(x'\right)\partial_a b^*\left(x'\right)}{\left[\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu\right]^2}d\mu\int\partial_{bb}g\left(x\right)\left[\partial_a b^*\left(x\right)1_{x=x''}+\partial_\lambda b^*\left(x\right)\partial_{a(x'')}\lambda\right]d\mu \\
&+\frac{\partial_a g\left(x'\right)+\partial_b g\left(x'\right)\partial_a b^*\left(x'\right)}{\left[\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu\right]^2}d\mu\int\partial_b g\left(x\right)\partial_{\lambda\lambda}b^*\left(x\right)\partial_{a(x'')}\lambda d\mu
\end{aligned}$$

Importantly, the limits of both the first and second-derivative can be expressed as

$$\begin{aligned}
\lim_{x\to\mathbb{E}[x]}\partial_{a(x')}\lambda^* &= \zeta\left[\mathbb{E}\left[x\right]\right]d\mu\left(x'\right) \\
\lim_{x\to\mathbb{E}[x]}\partial_{a(x')a(x'')}\lambda^* &= \xi\left[\mathbb{E}\left[x\right]\right]d\mu\left(x'\right)d\mu\left(x''\right)+1_{x'=x''}\psi\left[\mathbb{E}\left[x\right]\right]d\mu\left(x'\right)
\end{aligned}$$

for scalar functionals ζ,ξ,ψ. Plugging this into the equation above, we get

$$\lim_{x\to\mathbb{E}[x]}\frac{\delta^2 S_b}{\delta\hat{a}\left(x'\right)\delta\hat{a}\left(x''\right)}\left[a,x\right]=A\left[\mathbb{E}\left[x\right]\right]1_{x'=x''}d\mu\left(x'\right)+B\left[\mathbb{E}\left[x\right]\right]d\mu\left(x'\right)d\mu\left(x''\right)$$

for some functionals A and B, where

$$\begin{aligned}
A\left[x\right]=\partial_{aa}\tilde{f}\left(x\right)&-\frac{\int\partial_\lambda\tilde{f}d\mu}{\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu}\left[\partial_{aa}g\left(x\right)+2\partial_{ab}g\left(x\right)\partial_a b^*\left(x\right)\right] \\
&-\frac{\int\partial_\lambda\tilde{f}d\mu}{\int\partial_b g\left(x\right)\partial_\lambda b^*\left(x\right)d\mu}\left[\partial_{bb}g\left(x\right)\partial_a b^*\left(x\right)^2+\partial_b g\left(x\right)\partial_{aa}b^*\left(x\right)\right].
\end{aligned}$$

and

$$\partial_{aa}\tilde{f}\left(x\right)=\partial_{aa}f\left(x\right)+2\partial_{ab}f\left(x\right)\partial_a b^*\left(x\right)+\partial_{bb}f\left(x\right)\left(\partial_a b^*\left(x\right)\right)^2+\partial_b f\left(x\right)\partial_{aa}b^*\left(x\right)$$

After regrouping the terms, substituting $\partial_{aa}\tilde{f}$ and $\partial_\lambda\tilde{f}$, an application of the Dominated

Convergence Theorem implies that the limit is given by

$$
\begin{aligned}
A\left[\mathbb{E}\left[x\right]\right] = \partial_{aa} f\left(\mathbb{E}\left[x\right]\right) - \frac{\partial_b f\left(\mathbb{E}\left[x\right]\right)}{\partial_b g\left(\mathbb{E}\left[x\right]\right)} \partial_{aa} g\left(\mathbb{E}\left[x\right]\right) \\
+ 2\left(\partial_{ab} f\left(\mathbb{E}\left[x\right]\right) - \frac{\partial_b f\left(\mathbb{E}\left[x\right]\right)}{\partial_b g\left(\mathbb{E}\left[x\right]\right)} \partial_{ab} g\left(\mathbb{E}\left[x\right]\right)\right) \partial_a b^*\left(\mathbb{E}\left[x\right]\right) \\
+ \left(\partial_{bb} f\left(\mathbb{E}\left[x\right]\right) - \frac{\partial_b f\left(\mathbb{E}\left[x\right]\right)}{\partial_b g\left(\mathbb{E}\left[x\right]\right)} \partial_{bb} g\left(\mathbb{E}\left[x\right]\right)\right) \left(\partial_a b^*\left(\mathbb{E}\left[x\right]\right)\right)^2 .
\end{aligned}
$$

Thus, we can rewrite

$$
\begin{aligned}
S_b\left[a, x\right] - S_b\left[a^*, x\right] &= \frac{1}{2} A\left[\mathbb{E}\left[x\right]\right] \int_\Omega \int_\Omega 1_{x'=x''} \Delta\hat{a}\left(x'\right) \Delta\hat{a}\left(x''\right) d\mu\left(x'\right) \\
&\quad + \frac{1}{2} B\left[\mathbb{E}\left[x\right]\right] \int_\Omega \int_\Omega \Delta\hat{a}\left(x'\right) \Delta\hat{a}\left(x''\right) d\mu\left(x'\right) d\mu\left(x''\right) + o\left(\|x - \mathbb{E}\left[x\right]\|^2\right) \\
&= \frac{1}{2} A\left[\mathbb{E}\left[x\right]\right] \int_\Omega \Delta\hat{a}\left(x'\right)^2 d\mu\left(x'\right) + o\left(\|x - \mathbb{E}\left[x\right]\|^2\right) \\
&= \frac{1}{2} A\left[\mathbb{E}\left[x\right]\right] \mathbb{E}\left[\Delta a\left(x\right)^2\right] + o\left(\|x - \mathbb{E}\left[x\right]\|^2\right)
\end{aligned}
$$

where the latter terms grouped with B drop out due to centering. The first-order conditions of this program along a_{x_l} therefore characterize the second-order solution,

$$
\begin{aligned}
0 &= \mathbb{E}\left[\left\{a_0 + a_{x_l}^\intercal x_l - \left(a^*\left(\mathbb{E}\left[x\right]\right) + \partial_x a^*\left(\mathbb{E}\left[x\right]\right)^\intercal \left(x - \mathbb{E}\left[x\right]\right)\right)\right\} x_l\right] \\
0 &= \mathbb{E}\left[a_0 + a_{x_l}^\intercal x_l - \left(a^*\left(\mathbb{E}\left[x\right]\right) + \partial_x a^*\left(\mathbb{E}\left[x\right]\right)^\intercal \left(x - \mathbb{E}\left[x\right]\right)\right)\right] .
\end{aligned}
$$

From the second equation, it follows that $a\left(\mathbb{E}\left[x\right]\right) = a^*\left(\mathbb{E}\left[x\right]\right)$, verifying the initial conjecture. Thus, when $A\left[\mathbb{E}\left[x\right]\right] < 0$, the problem reduces to minimizing the tracking error (or maximizing it in the converse)

$$
\begin{aligned}
S_b\left[a\right] - S_b\left[a^*\right] &\propto -\mathbb{E}\left[\left\{a_{x_l}^\intercal \left(x_l - \mathbb{E}\left[x_l\right]\right) - \partial_x a^*\left(\mathbb{E}\left[x\right]\right)^\intercal \left(x - \mathbb{E}\left[x\right]\right)\right\}^2\right] \\
&= -a_{x_l}^\intercal \text{Var}\ \left(x_l\right) a_{x_l} - \partial_x a^*\left(\mathbb{E}\left[x\right]\right)^\intercal \text{Var}\ \left(x\right) \partial_x a^*\left(\mathbb{E}\left[x\right]\right) + 2 a_{x_l}^\intercal \text{Cov}\ \left(x_l, x\right) \partial_x a^*\left(\mathbb{E}\left[x\right]\right)
\end{aligned}
$$

which has the optimal solution of

$$
a_{x_l} = \text{Var}\ \left(x_l\right)^{-1} \text{Cov}\ \left(x_l, x\right) \partial_x a^*\left(\mathbb{E}\left[x\right]\right) .
$$

□

From this, the following Projection corollary follows.

Corollary 2. *Suppose the profit function satisfies $\partial_{PP}\pi \leq 0$, $\partial_{PQ}\pi \geq 1$, $\partial_{QQ}\pi \leq 0$.*

Let $P^*(x)$ denote the flexible pricing solution. As $x \to \mathbb{E}[x]$, to a second-order approximation the seller's optimal currency choice is the unconditional passthrough of exchange rates onto flexible prices,

$$\beta^* = \text{Var}\,(s)^{-1}\,\text{Cov}\,(s,x)\,\partial_x \bar{P}^*\left(\mathbb{E}[x]\right).$$

Proof. Changing the notation, note that the control is the price P which corresponds to action a as defined above. The substituted quantity menu Q corresponds to b, as defined above. The incentive and participation constraints are analytical and the willingness to pay has the codomain of the positive real line, so Lagrange multipliers exist. Both a participation and ex-post incentive constraint satisfy the representation of G in Lemma 1. The control is also multilinear as $P(s) = P_0 + \beta \cdot s$.

What remains to be shown is that the second-order condition is satisfied, that is to say that $A(\mathbb{E}[x]) \leq 0$, where A is defined in Lemma 2. Careful algebra reveals that

$$\begin{aligned} A\left(\mathbb{E}[x]\right) = M^i \partial_{PP}\pi + 2M^i \left(\partial_{PQ}\pi\left(\mathbb{E}[x]\right) - \frac{\partial_Q \pi\left(\mathbb{E}[x]\right)}{\bar{V} - \bar{P}}\right) \partial_P Q^*\left(\mathbb{E}[P], x\right) \\ + M^i \partial_{QQ}\pi\left(\mathbb{E}[x]\right)\left(\partial_P Q^*\left(\mathbb{E}[P], x\right)\right)^2 \end{aligned}$$

Note that the SDF is strictly positive. The first term is assumed to be negative. The second term follows by limited liability and the fact that $\partial_P Q \leq 0$. The third term is weakly negative by assumption. Consequently, $A(\mathbb{E}[x]) \leq 0$, completing the proof. □

A.1.3 Proof of Theorem 1

Proof. The optimal flexible price is given by

$$\left(M^i\left(\partial_P \pi + \partial_Q \pi \partial_P Q\right) + M^j \mathbb{E}\left[M^i \partial_Q \pi \partial_{\mathbb{E}[M^j P]} Q\right]\right)\mu(x) = 0 \qquad \forall x \in X.$$

Linearizing this yields

$$\begin{aligned} o\left(\|x - \mathbb{E}[x]\|\right) = \bar{\pi}_P\left(M^i - \bar{M}^i\right) + \bar{M}^i \bar{\pi}_{PP} \partial_x \bar{P}^*\left(x - \bar{x}\right) \\ + \bar{M}^i \bar{\pi}_{Px}\left(x - \bar{x}\right) + \bar{M}^i \partial_Q \bar{\pi} \partial_{\mathbb{E}[M^j P]} \bar{Q}\left(M^j - \bar{M}^j\right) + \text{constants}. \end{aligned}$$

Using the integrated FOC $\bar{M}^i \bar{\pi}_P = -\bar{M}^j \bar{M}^i \partial_Q \bar{\pi} \partial_{\mathbb{E}[M^j P]} \bar{Q}$,

$$\partial_x \bar{P}^*\left(x - \bar{x}\right) \approx -\left(\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}}\left(x - \bar{x}\right) + \frac{\bar{\pi}_P}{\bar{\pi}_{PP}}\left(\frac{M^i}{\bar{M}^i} - \frac{M^j}{\bar{M}^j}\right)\right) + \text{constants}.$$

It can be verified that $\bar{\pi}_P = \delta \partial_P \bar{\pi}$ in the seller's problem. Substitute in Definition 2. Applying Corollary 2 completes the proof. □

A.1.4 Proof of Proposition 2

Lemma 3. *Let $x \to \mathbb{E}[x]$. Suppose $\beta \in \mathcal{B}$. The optimal currency choice β minimizes the tracking error to the optimal FX passthrough*

$$\beta \in \arg\min_{\bar{\beta} \in \mathcal{B}} \left(\bar{\beta} - \beta^*\right)^\intercal \Sigma \left(\bar{\beta} - \beta^*\right).$$

where Σ is the covariance matrix of exchange rates and β^* is defined in Corollary 2.

Proof. Since P_0 is a control, we know that for any choice of β, the expected price must be centered. Thus, the objective is proportional to the tracking error of the flexible pricing solution $P^*(x)$.

$$\begin{aligned}
&\mathbb{E}\left[\left\{\beta^\intercal (s - \mathbb{E}[s]) - \left(\partial_x \left(\bar{P}^* (\mathbb{E}[x])\right)^\intercal (x - \mathbb{E}[x])\right)\right\}^2\right] \\
=&\mathbb{E}\left[\left\{(\beta - \beta^* + \beta^*)^\intercal (s - \mathbb{E}[s]) - \left(\partial_x \left(\bar{P}^* (\mathbb{E}[x])\right)^\intercal (x - \mathbb{E}[x])\right)\right\}^2\right] \\
=&(\beta - \beta^* + \beta^*)^\intercal \operatorname{Var}(s) (\beta - \beta^* + \beta^*) + \partial_x \bar{P}^* (\mathbb{E}[x])^\intercal \operatorname{Var}(x) \partial_x \bar{P}^* (\mathbb{E}[x]) \\
&\qquad - 2 (\beta - \beta^* + \beta^*) \operatorname{Cov}(s, x) \partial_x \bar{P}^* (\mathbb{E}[x]) \\
=&(\beta - \beta^*)^\intercal \operatorname{Var}(s) (\beta - \beta^*) + 2 (\beta - \beta^*)^\intercal \operatorname{Var}(s) (\beta^*) + (\beta^*)^\intercal \operatorname{Var}(s) \beta^* \\
&\qquad - 2 (\beta - \beta^*)^\intercal \operatorname{Cov}(s, x) \partial_x \bar{P}^* (\mathbb{E}[x]) - 2 (\beta^*)^\intercal \operatorname{Cov}(s, x) \partial_x \bar{P}^* (\mathbb{E}[x])
\end{aligned}$$

since β is the control, we can rewrite this as proportional to

$$\propto (\beta - \beta^*)^\intercal \left[\operatorname{Var}(s) (\beta - \beta^*) + 2\operatorname{Var}(s) \beta^* - 2\operatorname{Cov}(s, x) \partial_x \bar{P}^* (\mathbb{E}[x])\right]$$

using the condition

$$\beta^* = \operatorname{Var}(s)^{-1} \operatorname{Cov}(s, x) \partial_x \bar{P}^* (\mathbb{E}[x])$$

we get

$$\propto (\beta - \beta^*)^\intercal \operatorname{Var}(s) (\beta - \beta^*).$$

□

From this, the classic binary choice solution $n = 1$ is immediate.

Corollary 3. *For the PCP $\beta/P_0 = 0$ vs LCP $\beta/P_0 = 1$ discrete choice problem, to a second-order approximation, the threshold rule is*

$$\beta/P_0 = \begin{cases} 0 & \beta^*/P_0 < 1/2 \\ 1 & o.w. \end{cases}, \qquad as\ x \to \mathbb{E}[x]$$

Corollary 4. *For the n currency discrete choice problem, if the covariance matrix is a diagonal matrix with constant variance, the threshold rule minimizes the Euclidean distance.*

A.1.5 Equivalence to Forward Contracts

While the 1/2 threshold rule applies to the case where $\delta < 1$, it fails when quantities are entirely fixed $\delta = 1$.[1] In this context, I formalize an equivalence between currency choice in goods trade and in the forward exchange rate hedging problem.

I define the forward pricing problem as a setting in which a market participant (seller i) writes a forward contract with a market maker (buyer j), locking in a future exchange rate $\hat{F}$ for the transfer $\beta S(\omega)$. The proposed forward price must at least meet the reservation value of the market maker, giving rise to a participation constraint. And as before, the seller i exercises all the bargaining power.

Definition 7. The **forward pricing problem** is a currency denomination $\beta \in [0,1]$ and forward price $F \in \mathbb{R}_{++}$ that maximizes a market participant's ex-ante surplus, subject to market maker participation constraint:

$$\begin{aligned} (\beta, F) \in \arg\max_{\hat{\beta},\hat{F}} \mathbb{E}\left[M^i\left(\hat{\beta}S - \hat{F}\right)\right] \\ \text{s.t. } 0 \leq \mathbb{E}\left[M^j\left(\hat{F} - \hat{\beta}S\right)\right]. \end{aligned}$$

The forward pricing problem exists for the sole purpose of financial hedging. Currencies do not facilitate real trade in this setting since ex-post transfers net out to zero

$$\underbrace{\hat{\beta}S(\omega) - \hat{F}_0}_{\text{Mkt Participant Transfers}} + \underbrace{\hat{F}_0 - \hat{\beta}S(\omega)}_{\text{Mkt Maker Transfers}} = 0 \qquad \forall \omega \in \Omega.$$

Nonetheless, the market participant captures ex-ante surplus $\mathcal{S}$. Normalizing the participation constraint by $\mathbb{E}[M^j]$ and the participant's objective by $\mathbb{E}[M^i]$, one can net out the

[1]The flexible pricing solution becomes bang-bang, violating both the second-order condition and the continuity assumption.

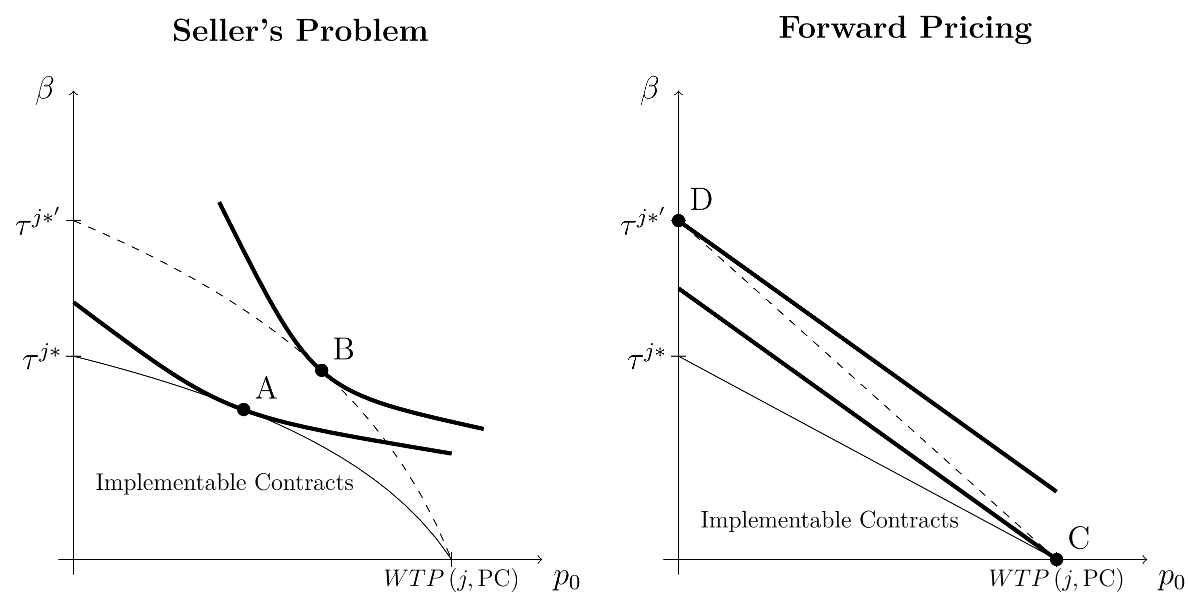

Figure 5: An increase in the buyer's local currency preference $\tau^{j*} \to \tau^{j*'}$ creates room for risk sharing. In the seller's problem, this changes currency denomination β from $A \to B$ and for forward pricing from $C \to D$.

transfer to demonstrate that the surplus is proportional to the total quantity of risk sharing

$$\mathcal{S} \propto \hat{\beta}\mathbb{E}\left[\left(\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}\right) S\right] = \hat{\beta}\Delta_{ij}F.$$

In the forward pricing problem, optimal currency choice $\hat{\beta} \in [0,1]$ is a corner solution that maximizes the surplus extracted from the relative cost of FX hedging $\Delta_{ij}F$. This stands in contrast to the real hedging theory, which features an interior currency choice that minimizes the tracking error between the realized and flexible monopoly price.

Lemma 4. *In the forward pricing problem, the optimal currency denomination satisfies*

$$\beta = \begin{cases} 0 & \Delta_{ij}F < 0 \\ 1 & \Delta_{ij}F \geq 0 \end{cases}.$$

The visual intuition for the seller's problem and forward pricing problem is characterized by Figure 5. In this figure, a seller chooses between producer and local currency pricing. The set of implementable price-quantity pairs is given by the region under the curve, with the x-axis representing the buyer's willingness to pay in producer terms and the y-axis representing the currency denomination scalar β. The bolded lines represent ex-ante seller surplus curves.

In the seller's problem the optimal currency share is given by A. A is on the frontier of implementable contracts satisfying the participation constraint. An increase in the seller's preference for the local currency shifts the implementable contract region upwards, leading to an increased local currency share given by B. The response is tempered by the real hedging incentive, reflected in convex shape of the seller's surplus curves in bold. In the forward pricing problem, the real hedging consideration disappears. The response becomes dramatic causing a total shift in the currency share from C to D.

The full contracting problem becomes identical to the forward pricing problem when quantities become fixed.

Proposition 6. *Let profits satisfy $\partial_P \pi = Q$. The optimal currency choice of the fixed quantity problem $\delta = 1$ is equivalent to that of the forward pricing solution, as in Lemma 4.*

Proof. The seller's problem is given by

$$\max \mathbb{E}\left[M^i \pi\left(P, Q, x\right)\right] \qquad \text{s.t. } Q = \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right).$$

The perturbation of β yields the effect

$$\begin{aligned}&\mathbb{E}\left[M^i\left(\partial_P \pi s + \partial_Q \pi \partial_{\mathbb{E}[M^j P]} \bar{V}^{-1} \mathbb{E}\left[M^j s\right]\right)\right]\\ =&\mathbb{E}\left[M^i Q s\right] + \mathbb{E}\left[M^i \partial_Q \pi \partial_{\mathbb{E}[M^j P]} \bar{V}^{-1}\right] \mathbb{E}\left[M^j s\right]\end{aligned}$$

using the fact that the perturbation of P_0 implies

$$\mathbb{E}\left[M^i Q\right] + \mathbb{E}\left[M^i \partial_Q \pi \partial_{\mathbb{E}[M^j P]} \bar{V}^{-1}\right] \mathbb{E}\left[M^j\right] = 0$$

we can rewrite the effect of the perturbation as

$$= \mathbb{E}\left[M^i s\right] - \frac{\mathbb{E}\left[M^i\right]}{\mathbb{E}\left[M^j\right]} \mathbb{E}\left[M^j s\right] \propto \Delta_{ij} F.$$

□

A.2 Details of the Armington Trade Model

Assume the profit function is

$$\pi\left(P, Q, x\right) = PQ - CQ$$

and the demand curves satisfy

$$V\left(Q, x\right) = \mathcal{P}\left(Q/\mathcal{Q}\right)^{-1/\sigma}.$$

V satisfies two identities. Two relationships must hold

$$V_{(x)}^{-1}(P) = \mathcal{Q}\left(P/\mathcal{P}\right)^{-\sigma}; \qquad \Rightarrow \partial_P V_{(x)}^{-1}(P) = -\sigma \frac{V_{(x)}^{-1}(P)}{P}$$

and second, using the inverse function theorem,

$$\partial_{\mathbb{E}[M^j P]} \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right) = \frac{1}{\mathbb{E}\left[M^j V_Q\left(\bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right), x\right)\right]}.$$

To characterize optimal currency choice, we start by characterizing the flexible price. The flexible price corresponds to the expected price of the nominal rigid solution. The flexible price satisfies the first-order condition

$$M^i\left(\partial_P \pi + \partial_Q \pi \partial_P Q\right) + M^j \mathbb{E}\left[M^i \partial_Q \pi \partial_{\mathbb{E}[M^j P]} Q\right] = 0.$$

Using the fact that

$$Q = (1-\delta) V_{(x)}^{-1}(P) + \delta \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right)$$

we can substitute in with

$$M^i\left(Q - \sigma(P - C)(1-\delta)\frac{V_{(x)}^{-1}(P)}{P}\right) - \sigma \frac{\delta M^j \mathbb{E}\left[M^i (P - C)\right]}{\mathbb{E}\left[M^j \frac{V\left(\bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right), x\right)}{\bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right)}\right]} = 0.$$

Taking the expected state $x(\omega) \to \bar{x}$, we get

$$\bar{M}^i\left(\bar{Q} - \sigma\left(1 - \bar{C}/\bar{P}\right)(1-\delta)\bar{Q}^m\right) - \sigma\delta \frac{\bar{M}^j \bar{M}^i\left(\bar{P} - \bar{C}\right)}{\bar{M}^j \bar{P}} \bar{Q}_0 = 0$$

using the fact that $\lim_{x \to \bar{x}} V_{(x)}^{-1}(P) = \lim_{x \to \bar{x}} \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right) = \bar{Q}$, we get

$$0 = \bar{Q} - \sigma\left(1 - \bar{C}/\bar{P}\right)\bar{Q}$$

implying that the expected flexible price satisfies

$$\bar{P} = \sigma\left(\bar{P} - \bar{C}\right) \Rightarrow \bar{P}^* = \frac{\sigma}{\sigma - 1}\bar{C}.$$

Given the standard solution, we now characterize the coefficients $\bar{\pi}_P$, $\bar{\pi}_{PP}$, and $\bar{\pi}_{Px}$ as these show up in the currency denomination formula. We have that

$$\begin{aligned}\bar{\pi}_P &= \bar{Q} - \sigma\left(1 - \bar{C}/\bar{P}\right)(1-\delta)\,\bar{Q} \\ &= \bar{Q}\left(1 - \sigma\left(1 - \bar{C}/\bar{P}\right)\right) + \delta\sigma\left(1 - \bar{C}/\bar{P}\right)\bar{Q}\end{aligned}$$

using the first order condition we can rewrite

$$\bar{\pi}_P = -\delta\sigma\left(1 - \bar{C}/\bar{P}\right)\bar{Q} = \delta\bar{Q}.$$

For the second-order condition, using $\pi_P = Q - \sigma\left(1 - \frac{C}{P}\right)(1-\delta)\,V_{(x)}^{-1}(P)$ and the definition of the full derivative, we get

$$\begin{aligned}\pi_{PP} &= \partial_P \pi_P + \partial_Q \pi_P \partial_P Q \\ &= \sigma^2\left(1 - \frac{C}{P}\right)(1-\delta)\,\frac{V_{(x)}^{-1}(P)}{P} - \sigma\frac{C}{P^2}(1-\delta)\,V_{(x)}^{-1}(P) + \partial_P Q \\ &= \sigma^2\left(1 - \frac{C}{P}\right)(1-\delta)\,\frac{V_{(x)}^{-1}(P)}{P} - (1-\delta)\,\sigma\frac{V_{(x)}^{-1}(P)}{P} \\ &\quad - \sigma\frac{C}{P^2}(1-\delta)\,V_{(x)}^{-1}(P)\end{aligned}$$

in the expected state, the top line drops out when we substitute in the optimal price. So we are left with,

$$\bar{\pi}_{PP} = -(\sigma - 1)(1-\delta)\,\bar{Q}/\bar{P}.$$

Moreover,

$$\begin{aligned}\pi_{Px} &= \partial_x\left[Q - \sigma\left(1 - \frac{C}{P}\right)(1-\delta)\,V_{(x)}^{-1}(P)\right] \\ &\quad + \partial_Q\left[Q - \sigma\left(1 - \frac{C}{P}\right)(1-\delta)\,V_{(x)}^{-1}(P)\right]\partial_x Q \\ &= -\sigma\left(1 - \frac{C}{P}\right)(1-\delta)\,\partial_x V_{(x)}^{-1}(P) \\ &\quad + \sigma\frac{\partial_x C}{P}(1-\delta)\,V_{(x)}^{-1}(P) + \partial_x Q \\ &= (1-\delta)\left(1 - \sigma + \sigma\frac{C}{P}\right)\partial_x V_{(x)}^{-1}(P) \\ &\quad + \sigma\frac{\partial_x C}{P}(1-\delta)\,V_{(x)}^{-1}(P)\end{aligned}$$

once again, the top line drops out in the limit. So we get,

$$\bar{\pi}_{Px} = \sigma \partial_x C \left(1 - \delta\right) \bar{Q} / \bar{P}.$$

Assembling the pieces, the steady state coefficients become

$$-\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} = \frac{\sigma}{\sigma - 1} \partial_x \bar{C} = \frac{\bar{P}}{\bar{C}} \partial_x \bar{C}$$

and

$$-\frac{\bar{\pi}_{P}}{\bar{\pi}_{PP}} = \frac{\delta \bar{Q}}{\left(\sigma - 1\right)\left(1 - \delta\right) \bar{Q} / \bar{P}} = \frac{\delta}{1 - \delta} \frac{1}{\sigma - 1} \bar{P}$$

thus

$$\beta^*_\delta / \bar{P} \approx \frac{\partial_x \bar{C}}{\bar{C}} b_{xs} + \frac{\delta}{1 - \delta} \frac{1}{\sigma - 1} \frac{\Delta_{ij} F}{\text{Var}\ (s)}.$$

Combining with the identity $\mathcal{M} = \frac{\sigma}{\sigma - 1}$, $P_0 = \bar{P}$, and using the fact that marginal costs only depend on exchange rates with passthrough γ, we get

$$\beta^*_\delta / P_0 \approx \gamma + \frac{\delta}{1 - \delta} \left(\mathcal{M} - 1\right) \frac{\Delta_{ij} F}{\text{Var}\ (s)}.$$

A.3 Example with Quantity Restrictions

In this section, we consider a simple example where the exporter needs to meet a minimum quantity in order to participate in the trade contract, often referred to a "minimum order quantity" rule. This helps demonstrate how the solution technique can be widely applied outside of a fixed quantity setting and captures a basic feature of many trade contracts. The optimal currency choice formula is broadly unchanged.[2]

As before, we assume the profit function is standard, i.e.

$$\pi\left(P, Q, x\right) = PQ - CQ$$

with the participation constraint given by

$$\mathbb{E}\left[M^j \left(V\left(Q, x\right) Q - PQ\right)\right] \geq 0$$

in addition to the constraint that

$$Q \geq \underline{Q}$$

[2]Special thanks to Omar Barbiero for suggesting this extension.

where in this equation, $\underline{Q}$ is some exogenous scalar that represents the minimum quantity exporter. As before, we specify the CES valuation curve $V(Q,x) = \mathcal{P}(Q/\mathcal{Q})^{-1/\sigma}$.

Intuitively, as exchange rates increase, the importer has a desire to reduce their demand for the foreign good. However, because of this minimum quantity restriction, the firm is unable to remain naturally hedged when exchange rates appreciate dramatically. Consequently, this introduces financial risk and commitment in the trade contract. Commitment occurs before if the quantity falls below a certain point, the importer will not receive any goods and the seller will not receive any profits. This creates an incentive for both parties to commit to at least delivering $\underline{Q}$ even when exchange rates appreciate ex-post.

To keep things tractable, we analyze the case where when quantities exceed the lower bound $Q > \underline{Q}$, the realized demand is given by the buyer's inverse valuation curve. This would capture the case where commitment only exists to ensure that trade is at least implemented when the quantity restriction is binding. Consequently, quantities can be expressed as

$$Q(x) = \max\left\{\underline{Q}, \mathcal{Q}\left(P(s)/\mathcal{P}\right)^{-\sigma}\right\}$$

where in this equation, the quantity constraint is always respected. Because the inverse valuation function has been substituted into quantities, it follows that we can partial out the participation constraint to only states where the restriction is binding

$$\mathbb{E}\left[M^j V\left(\underline{Q}, x\right) \mid Q = \underline{Q}\right] = \mathbb{E}\left[M^j P(s) \mid Q = \underline{Q}\right].$$

Notably, the more general case (where quantities are committed across all states) can be analyzed under the contracting framework detailed later in the Appendix.

Thus, the seller's problem is given by

$$\int M^i (P - C)\, \underline{Q}\mathbf{1}\left\{P \geq \mathcal{P}\left(\underline{Q}/\mathcal{Q}\right)^{-1/\sigma}\right\} d\mu$$
$$+ \int M^i (P - C)\, \mathcal{Q}\left(P/\mathcal{P}\right)^{-\sigma} \mathbf{1}\left\{P < \mathcal{P}\left(\underline{Q}/\mathcal{Q}\right)^{-1/\sigma}\right\} d\mu$$

where in the two parts, I have substituted in whether or not the price has resulted in quantities reaching the minimum threshold.

Taking the first-order condition with respect to the flexible price $P(x)$, we arrive at two first order conditions. The first is the usual monopoly pricing first-order condition, which holds when the quantity restriction does not bind $Q > \underline{Q}$:

$$1 - \sigma\left(1 - \frac{C}{P}\right) = 0$$

which implies that in these states, $P\left(x\right)=\frac{\sigma}{\sigma-1}C$. However, when the minimum quantity constraint binds, instead the first-order condition is defined by

$$M^{i}+\xi=\eta M^{j}$$

where in this equation, η is the associated Lagrange multiplier of the participation constraint, and ξ is the constraint that $P\left(x\right)\in\left[0,\bar{P}\right]$ ensures that $P\left(x\right)$ is a positive finite function. From this, it follows that we can rewrite the flexible price as a piecewise function:

$$P\left(x\right)=\begin{cases}\frac{\sigma}{\sigma-1}C\left(x\right) & Q>\underline{Q}\\ \bar{P} & Q=\underline{Q},M^{i}>M^{j}\\ 0 & Q=\underline{Q},M^{i}\leq M^{j}.\end{cases}$$

In this equation, the upper bound $\bar{P}$ is the scalar that satisfies $\bar{P}=\frac{\mathbb{E}\left[M^{j}V\left(\underline{Q},x\right)\mathbf{1}\left\{Q=\underline{Q}\right\}\right]}{\mathbb{E}\left[M^{j}\mathbf{1}\left\{Q=\underline{Q},M^{i}>M^{j}\right\}\right]}$. In other words, it is the maximum posted price that maximizes risk sharing while ensuring that the buyer is willing to commit to the trade contract.

To derive the optimal currency of invoicing, I now project the linearized flexible price onto the exchange rate. It follows from inspection that

$$\beta^{*}\approx\frac{\sigma}{\sigma-1}b_{cs}\mu\left(Q>\underline{Q}\right)+\bar{P}\left[\mathbb{E}[s\mid Q<\bar{Q},M^{i}>M^{j}]-\mathbb{E}[s]\right]\mu\left(Q<\underline{Q},M^{i}>M^{j}\right).$$

In this equation, the relative strength of the real hedging incentive and the financial hedging incentive are determined by the share of states where the seller and buyer are committed to delivering the minimum quantity. When $\underline{Q}=0$, i.e. there is no minimum quantity constraint, it follows by direct comparison that only real hedging matters.

In contrast, when the minimum quantity constraint always binds, the real hedging incentive drops out. The currency of invoicing choice targets whether or not the exchange rate tends to appreciate when the seller has a higher marginal utility of wealth than the buyer. This effect is fortified by the range of prices that the seller can implement, i.e. the size of $\bar{P}$. If $\bar{P}$ is very large, the currency of invoicing choice is bang-bang in financial hedging. Intuitively, this is because there is "scope" in the trade contract to charge higher prices in bad states for the seller, which increases the ex-ante level of risk sharing. In a more complete model, $\bar{P}$ would be pinned down by the availability of commitment or endogenous risk sharing, as eventually, an arbitrarily high price would likely cause the buyer to renegotiate the terms of trade.

A.4 Proof of Proposition 3

Denote λ_t as the variable Lagrange multipliers associated with the resource constraints (21) and (22). The planner's primal allocations of tradable consumption $\{C_{T,0}, C_{T,1}\}$ and investment $\left\{B_1^F, B_1^H\right\}$ are characterized by the following first-order conditions:

$$0 = \partial_{C_T} U_0 \left(1 + \frac{\alpha_0}{p_0}\tau_0^L\right) - P_{T,0}\lambda_0 \tag{26}$$

$$0 = \mu_1 \cdot \left[\partial_{C_T} U_1 \left(1 + \frac{\alpha_1}{p_1}\tau_1^L\right) - P_{T,1}\lambda_1\right] + \tau^S \frac{d\beta}{dC_{T,1}} \tag{27}$$

$$1 = \rho \mathbb{E}\left[\frac{\lambda_1}{\lambda_0} R_0^* S_1\right] \tag{28}$$

$$1 = \rho \mathbb{E}\left[\frac{\lambda_1}{\lambda_0} R_0^* \mathbb{E}\left[1/S_1\right]\right] \tag{29}$$

where τ^S can be verified as the social value of increasing β, holding the other controls fixed.

Combining Equations (26)-(29) yields the Euler equations of the social planner.

$$1 = \mathbb{E}\left[M_1 \frac{1 + \frac{\alpha_1}{p_1}\tau_1^L + \frac{d\beta/dC_{T,1}}{\mu_1 \partial_{C_T} U_1}\tau^S}{1 + \frac{\alpha_0}{p_0}\tau_0^L} R\right]$$

which can be combined with the competitive implementation for the dollar bond to yield the desired capital control

$$\begin{aligned} 1 + \tau^{B^F} &= \frac{\mathbb{E}\left[M_1 \frac{1 + \frac{\alpha_1}{p_1}\tau_1^L + \frac{d\beta/dC_{T,1}}{\mu_1 \partial_{C_T} U_1}\tau^S}{1 + \frac{\alpha_0}{p_0}\tau_0^L}\right]}{\mathbb{E}\left[M_1 S_1\right]} \\ &= \mathbb{E}\left[\frac{1 + \frac{\alpha_1}{p_1}\tau_1^L + \frac{d\beta/dC_{T,1}}{\mu_1 \partial_{C_T} U_1}\tau^S}{1 + \frac{\alpha_0}{p_0}\tau_0^L}\right] + \frac{\text{Cov}\left(M_1 S_1, \frac{1 + \frac{\alpha_1}{p_1}\tau_1^L + \frac{d\beta/dC_{T,1}}{\mu_1 \partial_{C_T} U_1}\tau^S}{1 + \frac{\alpha_0}{p_0}\tau_0^L}\right)}{\mathbb{E}\left[M_1 S_1\right]}. \end{aligned}$$

For the foreign tax on capital inflows, I compare the first order condition with the foreign investor's breakeven condition:

$$\begin{aligned} R_0^* \left(1 + \tau^{B^H}\right) &= R_0 \mathbb{E}\left[\frac{1}{S_1}\right] \\ &= \frac{\mathbb{E}\left[\frac{1}{S_1}\right]}{\mathbb{E}\left[M_1\right]} \end{aligned}$$

which can be divided through to yield the tax on home bond debt:

$$1+\tau^{B^H}=\mathbb{E}\left[\frac{1+\frac{\alpha_1}{p_1}\tau_1^L+\frac{d\beta/dC_{T,1}}{\mu_1\partial_{C_T}U_1}\tau^S}{1+\frac{\alpha_0}{p_0}\tau_0^L}\right]+\frac{\text{Cov}\left(M_1,\frac{1+\frac{\alpha_1}{p_1}\tau_1^L+\frac{d\beta/dC_{T,1}}{\mu_1\partial_{C_T}U_1}\tau^S}{1+\frac{\alpha_0}{p_0}\tau_0^L}\right)}{\mathbb{E}\left[M_1S_1\right]}.$$

Finally, all that remains is to derive $d\beta/dC_{T,1}$. Consider a perturbation of Equation (20) that holds the path of exchange rates constant, which is explicitly controlled by the planner,

$$\begin{aligned}\frac{d\beta}{\mu_1\partial_{C_T}U_1}&=-\mathbb{E}\left[P_{T,1}\right]\frac{\frac{\delta}{1-\delta}\left(\mathcal{M}-1\right)}{\text{Var}\left(S_1\right)\cdot\partial_{C_T}U_1}\left(\frac{dM_1}{\mathbb{E}\left[M_1\right]}S_1-\frac{\mathbb{E}\left[M_1S_1\right]}{\mathbb{E}\left[M_1\right]}\frac{dM_1}{\mathbb{E}\left[M_1\right]}\right)\\&=-\mathbb{E}\left[P_{T,1}\right]\frac{\frac{\delta}{1-\delta}\left(\mathcal{M}-1\right)}{\text{Var}\left(S_1\right)}\left(S_1-\mathbb{E}\left[\frac{M_1}{\mathbb{E}\left[M_1\right]}S_1\right]\right)\frac{dM_1}{\partial_{C_T}U_1\mathbb{E}\left[M_1\right]}\end{aligned}$$

Now I derive the remaining properties of this change in trade invoicing. I demonstrate that $dM/dC_{T,1}$ is the coefficient of relative risk aversion in a special case of the model. I compute the total effect of a perturbation in tradable consumption and combining the effect with the identity $\mathbb{E}\left[M_1\right]=\frac{1}{R_0}$. The perturbation jointly increases tradable consumption and non-tradable consumption, which is an implicit function of the tradable consumption due to the implementability constraints:

$$dM_1/dC_{T,1}=\rho\frac{\partial_{C_T}^2U_1+\alpha_1\partial_{C_T}\partial_{C_{NT}}U_1}{\partial_{C_T}U_0}.$$

Labor supply does not enter the equation because it is separable in the utility function. The perturbation has the following properties.

Lemma 5. *The perturbation in the SDF due to tradable consumption satisfies,*

$$\frac{dM_1/dC_{T,1}}{\partial_{C_T}U_1\mathbb{E}\left[M_1\right]}=\frac{\frac{\partial_C^2U_1\partial_{C_T}C}{\partial_CU_1}\left(1+\frac{\alpha_1}{p_1}\right)+\frac{\partial_{C_T}^2C}{\partial_{C_T}C}\left(1+\frac{\alpha_1}{p_1}\right)+\left(\frac{\alpha_1}{p_1}\right)^2\frac{1}{\partial_p\alpha_1C_T}}{\mathbb{E}\left[\partial_{C_T}U_1\right]}$$

and in the special case of Cobb-Douglas preferences over tradable and non-tradables reduces down to

$$\frac{dM_1/dC_{T,1}}{\partial_{C_T}U_1\mathbb{E}\left[M_1\right]}=-\frac{\gamma}{C_{T,1}\mathbb{E}\left[\partial_{C_T}U_1\right]}$$

where γ is the coefficient of relative risk aversion (which may be stochastic).

Proof. Using homotheticity over consumption, it follows that

$$\begin{aligned}
&\partial^2_{C_T} U_1 + \alpha_1 \partial_{C_T} \partial_{C_{NT}} U_1 \\
=&\partial^2_C U_1 \left(\partial_{C_T} C\right)^2 + \partial_C U_1 \partial^2_{C_T} C \\
&\qquad + \alpha_1 \partial^2_C U_1 \partial_{C_T} C \partial_{C_{NT}} C + \alpha_1 \partial_C U_1 \partial_{C_T} \partial_{C_{NT}} C
\end{aligned}$$

in addition, we know that a property of homothetic functions is a constant ratio in the marginal rate of substitution

$$\frac{\partial_{C_T} C}{\partial_{C_{NT}} C} = \phi\left(\frac{C_{NT}}{C_T}\right) = p_t$$

where p_t is the relative price. By definition, we know that $C_{NT} = \alpha_t C_T$, meaning $\phi = \alpha^{-1}$ and $\phi = p_t$ in any equilibrium. Substituting $\partial_{C_{NT}} U = \frac{1}{p_t} \partial_{C_T} U$ in,

$$\begin{aligned}
&\partial^2_C U_1 \left(\partial_{C_T} C\right)^2 \left(1 + \frac{\alpha_1}{p_1}\right) \\
&\qquad + \partial_C U_1 \partial^2_{C_T} C + \alpha_1 \partial_C U_1 \partial_{C_T} \partial_{C_{NT}} C.
\end{aligned}$$

Now differentiate the relationship implied by homotheticity,

$$\begin{aligned}
\partial_{C_T}\left(\partial_{C_{NT}} C\right) &= \partial_{C_T}\left(\frac{1}{\phi\left(\frac{C_{NT}}{C_T}\right)} \partial_{C_T} C\right) \\
&= \frac{1}{p_t} \partial^2_{C_T} C + \frac{\phi'}{\phi^2} \frac{C_{NT}}{C_T^2} \partial_{C_T} C
\end{aligned}$$

and combine with the inverse function theorem $\phi' = 1/\alpha'$,

$$\begin{aligned}
&\alpha_1 \partial_C U_1 \frac{1/\alpha'}{p_1^2} \frac{\alpha_1}{C_T} \partial_{C_T} C \\
=&\left(\frac{\alpha_1}{p_1}\right)^2 \frac{1}{\alpha_1' C_T} \partial_C U_1 \partial_{C_T} C.
\end{aligned}$$

Putting all the terms together and dividing through by $\partial_{C_T} U = \partial_C U_1 \partial_{C_T} C$, one derives the first result:

$$\frac{\partial^2_C U_1 \partial_{C_T} C}{\partial_C U_1}\left(1 + \frac{\alpha_1}{p_1}\right) + \frac{\partial^2_{C_T} C}{\partial_{C_T} C}\left(1 + \frac{\alpha_1}{p_1}\right) + \left(\frac{\alpha_1}{p_1}\right)^2 \frac{1}{\partial_p \alpha_1 C_T}.$$

Finally, consider the special case of Cobb-Douglas consumption. Without loss of generality, take $C_t = C_{NT,t}^{\chi} C_{T,t}^{1-\chi}$ where χ is the expenditure share on non-tradables. Using the Cobb-Douglas identity $\alpha_1 = p_1 \frac{\chi}{1-\chi}$ and $\partial_p \alpha_1 = \frac{\chi}{1-\chi}$. Combining and reducing terms, one arrives at

$$\frac{dM_1/dC_{T,1}}{\partial_{C_T} U_1 \mathbb{E}\left[M_1\right]} = \frac{\partial_C^2 U_1 C_1}{\partial_C U_1} \frac{1}{C_{T,1} \mathbb{E}\left[\partial_{C_T} U_1\right]}$$

which is the negative coefficient of relative risk aversion. □

A.5 Proof of Proposition 4

The problem is set up as

$$\begin{aligned}\mathcal{L} = \min_{\lambda \in C(X), \eta \geq 0} \max_{P,Q \in C(X)^2} \int M^i \pi \left(P, Q, x\right) + \eta M^j v \left(P, Q, x\right) d\mu \\ + \int \lambda \left(x\right) \left[v \left(P, Q, x\right) - \tilde{v} \left(P, x\right)\right] d\mu\end{aligned}$$

where the set of implementable deviations is reduced down to quantities by the nominal rigidity assumption $\tilde{v}\left(P\left(s\right), x\right) = \arg\max_{\hat{Q} \in \mathbb{D}_{jQ}^{rng}(x)} v\left(P\left(s\right), \hat{Q}, x\right)$. A set of Lagrange multipliers exist since the differential operator is defined for v and an optimal contract is assumed to exist.

Thus, we may take the first-order conditions of this with respect to $Q\left(x\right)$ as

$$M^i \partial_Q \pi \left(x\right) + \eta M^j \partial_Q v \left(x\right) + \lambda \left(x\right) \partial_Q v \left(x\right) = 0.$$

When $\lambda\left(x\right) > 0$, it follows that $v\left(P\left(s\right), Q\left(x\right), x\right) = \tilde{v}\left(P\left(s\right), x\right)$. By the implicit function theorem, define $Q^{rng}\left(P\left(s\right), x\right)$ to describe this relationship and note that it is of class C^∞.

Consequently, piece-wise we may express

$$Q^*\left(x\right) = \begin{cases} Q\left(P\left(s\right), x, \eta\right) & \lambda\left(x\right) = 0 \\ Q^{rng}\left(P\left(s\right), x\right) & \lambda\left(x\right) > 0 \end{cases}$$

where η is the scalar Lagrange multiplier on the participation constraint. The set of states $X^{cmt} := \{x \in X : v\left(P\left(s\right), Q\left(x\right), x\right) > \tilde{v}\left(P\left(s\right), x\right)\}$ can therefore be expressed as $X^{cmt} = \{x \in X : \lambda\left(x\right) = 0\}$. The forward direction of the proof is therefore immediate: if $\mu\left(X^{cmt}\right) = 0$, quantities are flexible.

To finish the proof, we show the reverse. Suppose for contradiction that $\mu\left(X^{cmt}\right) > 0$ and quantities are flexible. There is a positive measure of states such that Q is described by the implicit relationship

$$M^{i}\partial_{Q}\pi\left(x\right)+\eta M^{j}\partial_{Q}v\left(x\right)=0.$$

Since $\partial_{Q}\pi\left(x\right)>0$ and $M^{i}>0$ is a stochastic discount factor, it follows that $\eta M^{j}\partial_{Q}v\left(x\right)<0$. Consequently $\eta>0$ as $M^{j}>0$ and $\partial_{Q}v\left(x\right)<0$.

Thus the participation constraint holds for some neighborhood around Q^{*} and η^{*}.

$$\mathbb{E}\left[M^{j}\left[1_{\lambda=0}v\left(P,Q\left(P,x,\eta^{*}\right),x\right)+\left(1-1_{\lambda=0}\right)\tilde{v}\left(P,x\right)\right]\right]=0$$

the implicit function as applied to Banach spaces allows us to rewrite $\eta:C\left(X\right)\times C\left(\Omega\right)\mapsto\mathbb{R}_{+}$

$$\eta^{*}=\eta\left[P,x\right].$$

Consequently, Q is a functional of P and x, that is to say

$$Q^{*}\left(x\right)=\begin{cases}Q\left(P\left(s\right),x,\eta\left[P,x\right]\right) & \lambda\left(x\right)=0\\ Q^{rng}\left(P\left(s\right),x\right) & \lambda\left(x\right)>0\end{cases}.$$

Because $\lambda\left(x\right)=0$ happens with positive probability, it follows by contradiction that flexible quantities imply $\mu\left(X^{cmt}\right)=0$.

A.6 Proof of Proposition 5

Since Q^{*} satisfies the IC and PC, we may substitute out the constraints and directly maximize

$$\max_{P\in C(X)}\int_{\lambda=0}M^{i}\pi\left(P,Q\left(P,x,\eta\left[P,x\right]\right),x\right)d\mu+\int_{\lambda>0}M^{i}\pi\left(P,Q^{rng}\left(P,x\right),x\right).$$

I now characterize the optimal flexible price $P^{*}\left(x\right)$. The pointwise first order conditions for $P\left(x\right)$ are given by

$$M^{i}\left[\partial_{P}\pi+\partial_{Q}\pi\partial_{P}Q\right]+\mathbb{E}\left[M^{i}\partial_{Q}\pi\partial_{\eta}Q1_{\lambda=0}\right]\frac{\delta\eta}{\delta P}1_{\eta>0}=0$$

when $\lambda\left(x\right)=0$. Otherwise, when differentiating by $P\left(x\right)$ for $\lambda\left(x\right)>0$, we get

$$M^{i}\left[\partial_{P}\pi+\partial_{Q}\pi\partial_{P}Q^{rng}\right]1_{\lambda>0}=0.$$

The participation constraint meanwhile reads as

$$\eta\left[\int_{\lambda=0}M^{j}v\left(P\left(x\right),Q\left(x\right),x\right)d\mu+\int_{\lambda>0}M^{j}\tilde{v}\left(P\left(x\right),x\right)\right]=0.$$

To recover $\delta\eta/\delta P 1_{\eta>0}$, totally differentiate the constraint with $\lambda(x)=0$,

$$M^j\left[\partial_P v+\partial_Q v\partial_P Q\right]\int 1_x d\mu(x)+\mathbb{E}\left[M^j 1_{\lambda=0}\partial_Q v\partial_\eta Q\right]\frac{\delta\eta}{\delta P}=0$$

thus

$$\frac{\delta\eta}{\delta P}=-M^j\frac{\left[\partial_P v+\partial_Q v\partial_P Q\right]}{\mathbb{E}\left[M^j 1_{\lambda=0}\partial_Q v\partial_\eta Q\right]}\int 1_x d\mu(x)$$

and so we may rewrite

$$M^i\pi_P-M^j\left[\partial_P v+\partial_Q v\partial_P Q\right]\frac{\mathbb{E}\left[M^i 1_{\lambda=0}\partial_Q \pi\partial_\eta Q\right]}{\mathbb{E}\left[M^j 1_{\lambda=0}\partial_Q v\partial_\eta Q\right]}1_{\eta>0}=0.$$

Note that $\partial_P v+\partial_Q v\partial_P Q$ is the financial risk on v as defined in Definition 1. Thus

$$M^i\pi_P-M^j v_P\frac{\mathbb{E}\left[M^i 1_{\lambda=0}\partial_Q \pi\partial_\eta Q\right]}{\mathbb{E}\left[M^j 1_{\lambda=0}\partial_Q v\partial_\eta Q\right]}1_{\eta>0}=0.$$

Because $\mathbb{D}_j^{rng}$ is a continuous correspondence, the function $\tilde{v}(P(x),x)$ is analytic with respect to x. This allows us to apply Corollary 2 to analyze β^*. Taking $x\to\mathbb{E}[x]$,

$$\lim_{x\to\mathbb{E}[x]}\frac{\mathbb{E}\left[M^i 1_{\lambda=0}\partial_Q \pi\partial_\eta Q\right]}{\mathbb{E}\left[M^j 1_{\lambda=0}\partial_Q v\partial_\eta Q\right]}=\frac{\bar{M}^i\partial_Q\bar{\pi}\partial_\eta\bar{Q}}{\bar{M}^j\partial_Q\bar{v}\partial_\eta\bar{Q}}=\frac{\bar{M}^i\partial_Q\bar{\pi}}{\bar{M}^j\partial_Q\bar{v}}.$$

Linearizing this first-order condition, we get

$$\begin{aligned}o\left(\|x-\mathbb{E}[x]\|\right)=&\left(M^i-\bar{M}^i\right)\bar{\pi}_P+\bar{M}^i\bar{\pi}_{PP}\partial_x\bar{P}(x-\bar{x})+\bar{M}^i\bar{\pi}_{Px}(x-\bar{x})\\&-\left(M^j-\bar{M}^j\right)\bar{v}_P\frac{\bar{M}^i\partial_Q\bar{\pi}}{\bar{M}^j\partial_Q\bar{v}}1_{\eta>0}-\frac{\bar{M}^i\partial_Q\bar{\pi}}{\partial_Q\bar{v}}\bar{v}_{PP}1_{\eta>0}\partial_x\bar{P}(x-\bar{x})\\&-\frac{\bar{M}^i\partial_Q\bar{\pi}}{\partial_Q\bar{v}}\bar{v}_{Px}1_{\eta>0}(x-\bar{x})\end{aligned}$$

with the additional restriction $\bar{\pi}_P=\bar{v}_P\frac{\partial_Q\bar{\pi}}{\partial_Q\bar{v}}1_{\eta>0}$, this becomes

$$\begin{aligned}o\left(\|x-\mathbb{E}[x]\|\right)=&\left(\frac{M^i}{\bar{M}^i}-\frac{M^j}{\bar{M}^j}\right)\bar{\pi}_P+\left(\bar{\pi}_{PP}-\frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}\right)\partial_x\bar{P}(x-\bar{x})\\&+\left(\bar{\pi}_{Px}-\frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{Px}\right)(x-\bar{x}).\end{aligned}$$

The optimal currency of invoicing is the projection of the linearized flexible price onto exchange rates. By applying Corollary 2, one arrives at,

$$\beta^* \approx -\left(\underbrace{\frac{\bar{\pi}_{Px} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{Px}}{\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}} b_{xs}}_{\text{Real Hedging}} + \underbrace{\bar{\pi}_P \frac{\mu\left(X^{cmt}\right)\Sigma^{-1}\Delta_{ij|X^{cmt}}\tau}{\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}}}_{\text{Financial Hedging}}\right).$$

B The Financial Functions of a Currency

In this section, I show how my model, which assumes exogenous stochastic discount factors for buyers and sellers, captures standard mechanisms in the financial hedging literature. I demonstrate this by showing how the relative cost of FX hedging can be rewritten in terms of financial frictions such as uninsurable payment risk, trade financing costs, transaction taxes, and seller liquidity risk. I use these examples to revisit cross-country empirical patterns in currency invoicing that I interpret as evidence of financial hedging.

B.1 Buyer Payment Risk

Buyer payment risk is a common source of financial hedging in currency denomination (Doepke and Schneider, 2017; Drenik, Kirpalani and Perez, 2022). A theory of money characterizes currencies as a unit of account—so that, when the buyer is liable for payment, all else equal she prefers to pay in a currency that covaries with her wealth.

Consider a risk-neutral seller and a potentially risk-averse buyer. The risk-neutral seller has a constant discount factor $M^i = R^{-1}$. On the other hand, the risk averse buyer has an indirect utility function U which takes as arguments their wealth and the state of the world. The buyer invests in a producer and local currency deposit, earning interest rates R and R^*. Their share in the local currency deposit is given by θ, so that the maximization is written

$$\max_{\theta} \mathbb{E}\left[U\left(W, x\right)\right] \qquad \text{s.t. } W \leq W_0\left[R + \theta\left(R^* S - R\right)\right].$$

This objective models buyer payment risk. As the realized wealth W falls, the buyer's indirect utility can become arbitrarily negative due to an Inada condition. Denote the coefficient of relative risk aversion as $RRA := -\frac{\partial_{ww}U \cdot W}{\partial_w U}$.

Proposition 7. *To a second-order approximation, the buyer's optimal LC savings share is*

$$\theta \approx \frac{\mathbb{E}\left[R^* S - R\right]}{RRA \cdot \text{Var}\left(s\right)} \qquad \textit{as } x \to \mathbb{E}\left[x\right].$$

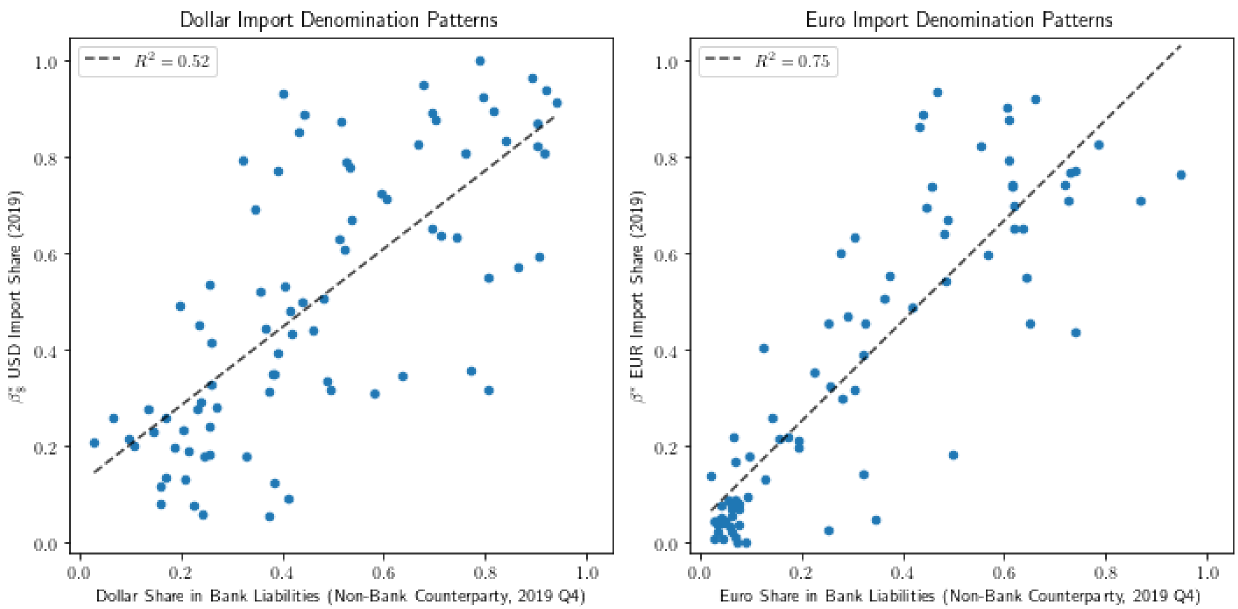

Figure 6: This scatter plot compares the cross-country currency denomination patterns versus the buyer's domestic banking sector foreign currency liabilities shares. The x-axis represents one measurement of financial hedging.

So a sufficient statistic for the relative cost of FX hedging is given by

$$\Delta_{ij} F := R^{-1} \cdot \underbrace{RRA \cdot \text{Var}\,(s)}_{Risk\ Aversion} \cdot \underbrace{\theta}_{LC\ Share} \qquad as\ x \to \mathbb{E}\,[x]\,.$$

Proof. See Appendix B.5. □

In this application of the theory, currency invoicing reflects the buyer's local currency wealth shares. Intuitively, the trade contract diversifies the buyer's investment risk and leads to an efficient price concession. This predicts a positive empirical relationship between a currency's import share and the buyer's cross-currency deposit shares.

The mapping between the theory and data is measurable through aggregate patterns. To the extent that trade contracts share financial risk, aggregate pricing patterns should reflect aggregate foreign currency debt investments. Aggregate import currency invoicing patterns are available for 70 countries in 2019 due to a dataset constructed by the World Bank (Boz et al., 2022). Meanwhile, buyer cross-currency investment shares can be measured using the BIS Location Banking Statistics, which report the quantity of bank liabilities in various currencies. Specifically, I focus on the liabilities where the counterparty is a nonbanking institution, for example capturing retail firms and households with preexisting deposits. There are substantial limitations to using BIS data since the foreign banking sector does not capture other debt securities that a buyer may purchase, such as corporate or sovereign debt. The representativeness of the BIS data therefore biases the relationship towards zero, and works against finding a pattern.

Figure 6 plots a positive relationship between currency denomination shares and the buyer's foreign currency banking deposit shares. This pattern suggests that buyer payment risk is relevant in the currency denomination of international goods trade. A larger Dollar and Euro share in the buyer's bank deposit sector suggests that more depositors hold their investments in foreign currency. Intuitively, the trade contract diversifies this investment risk by choosing the dollar and the euro. The relationship is strong but incomplete—theory anticipates that real hedging explains the residual variation in currency choice.

B.2 Trade Financing Costs

When the seller borrows across different currencies, cross-currency trade financing frictions measure financial hedging. Intuitively, currency choice in the trade contract alleviates financing frictions, since it serves as collateral against a seller's net debt position. This process is often referred to as "trade credit" and is vital in international goods trade (Bocola and Bornstein, 2023; Amiti and Weinstein, 2011; Ronci, 2004; Iacovone et al., 2019).

I now augment the model to include cross-currency financing. The exporter must borrow B to invest in capital I. They can borrow through their local bank that charges an upward sloping interest rate curve $R : \mathbb{R} \mapsto \mathbb{R}_{++}$ and $R^* : \mathbb{R} \mapsto \mathbb{R}_{++}$ with constant elasticity $\partial_b r \geq 0$ and $\partial_{b^*} r^* \geq 0$. The profit maximization problem is now

$$\max_{P_0, \beta^\intercal, B, I} \mathbb{E}\left[M^i \left(\underbrace{\pi\left(P, Q, I, x\right)}_{\text{Gross Income}} - \underbrace{R^*\left(B^*\right) S B^*}_{\text{Foreign CC}} - \underbrace{R\left(B\right) B}_{\text{Domestic CC}}\right)\right]$$

subject to the total level of borrowing at least exceeding the level of investment

$$B + B^* \geq I.$$

Otherwise, the stated problem is identical.

Modeling the cross-currency financing problem reveals a sufficient statistic for the value of financial hedging. Reading off the seller's first-order condition,

Proposition 8. *Assume the buyer's risk preferences are determined by the local bank,*

$$\mathbb{E}\left[\frac{M^j}{\mathbb{E}\left[M^j\right]} S\right] = F.$$

A sufficient statistic for the relative cost of FX hedging is given by

$$\Delta_{ij} F := \underbrace{\frac{1+\partial_b r}{1+\partial_{b^*} r^*}}_{\textit{Market Depth}} - \underbrace{F \cdot \frac{R^*}{R}}_{\textit{CIP}}.$$

Proof. See Appendix B.6. □

For each currency, two characteristics govern the degree of financial hedging: deviations in covered interest rate parity and relative market depth. Covered interest rate parity is the no-arbitrage condition that domestic interest rates R should equal foreign interest rates R^* after hedging exchange rate risk F.[3] In the model proposed by Gopinath and Stein (2021), covered interest rate parity is a sufficient statistic for the choice of currency, because it is precisely the shadow price of a costly collateral constraint. On the other hand, the relative market depth $\frac{1+\partial_b r}{1+\partial_{b^*} r^*}$ reflects the impact of sourcing capital across domestic and foreign capital markets. In the model proposed by Coppola, Krishnamurthy and Xu (2023), market depth is a sufficient statistic for currency choice, because it measures the monopsony rents collected from denominating a firm's balance sheet in a liquid currency.

Financial hedging is related to these two features through revealed preference. Intuitively, the size of these statistics reflects a financial friction that leads to an Euler equation wedge. Thus, the Euler equation wedge $\Delta_{ij} F$ can be measured by characteristics of the foreign exchange market, such as the size of CIP and market depth differentials. The characteristics then reveal the cost of financial frictions, such as unmodeled capital controls, search costs, taxes, leverage constraints, and collateral requirements, which force the firm to leave money on the table.

B.3 FX Transaction Costs

Another measurement of financial hedging is total transaction costs. Transaction costs come in the form of bid-ask spreads, creating a wedge between the hypothetical market exchange rate and the realized market exchange rate. Swoboda (1969), Krugman (1980), and Rey (2001) associate the theory of a dominant currency with transaction costs, which may arise due to search frictions as in Matsuyama, Kiyotaki and Matsui (1993).

This section formalizes the relationship between bid-ask spreads and financial hedging. A trade contract overcomes transaction costs, since the choice of currency denomination circumvents the buyer and seller going through a costly intermediary. To formally characterize

[3]Note that $S = 1$ so F is in fact the forward premium. In recent years, there has been evidence covered interest rate parity deviations due to banking regulation (Du, Tepper and Verdelhan, 2018; Du and Schreger, 2016; Keller, 2024).

transaction costs, the market exchange rate now depends on whether a currency is being bought or sold and which currency it is being converted to and from.

Definition 8. The bid-ask of an exchange rate S_c, for currency $c \in \mathcal{C}$, and for agents $a \in \{i, j\}$ are scalars satisfying

$$S_{ac}^{ask} := S_c\left(1 + \tau_{ac}^{ask}\right) \qquad S_{ac}^{bid} := S_c\left(1 + \tau_{ac}^{bid}\right) \qquad \forall a \in \{i, j\}, c \in \mathcal{C}$$

with the **bid-ask spread** defined as $\Delta_{bid/ask}\tau_{ac} := \frac{1+\tau_{ac}^{bid}}{1+\tau_{ac}^{ask}} \geq 1$.

The bid-ask spreads are a direct measure of transaction costs for a currency. τ_a is a percentage spread between the hypothetical market exchange rate S and agent a's quoted exchange rate S_a. Differences across agents $\tau_i \neq \tau_j$ reflect the fact that the seller i and buyer j may ultimately convert cashflows into different currencies or through different intermediaries. Differences across the bid and ask $\tau_a^{ask} \leq 0 \leq \tau_a^{bid}$ reflect the fact that the sale of a currency (ask) faces a markdown while its purchase (bid) faces a markup.

In the trade contract, the buyer receives exchange rate risk while the seller is short the exchange rate risk. This leads to two Euler equations which reflect the buyer and seller's risk preferences over quoted exchange rates.

Proposition 9. *For the transfer between buyer to seller, the Euler equation satisfies*

$$\underbrace{1 = \mathbb{E}\left[M^i \frac{S_{ic}^{bid\prime}}{S_{ic}^{ask}} R_c^*\right]}_{\text{Seller Long FX}}; \qquad \underbrace{1 = \mathbb{E}\left[M^i \frac{S_{ic}^{ask\prime}}{S_{ic}^{bid}} R_c^*\right]}_{\text{Buyer Short FX}}.$$

Assume the bid-ask spread is constant over time. A sufficient statistic for the relative cost of FX hedging is given by

$$\Delta_{ij}F := \underbrace{\frac{\Delta_{bid/ask}\tau_i}{\Delta_{bid/ask}\tau_i^*}}_{\text{Seller Net T-Costs}} - \underbrace{\frac{\Delta_{bid/ask}\tau_j^*}{\Delta_{bid/ask}\tau_j}}_{\text{Buyer Net T-Costs}}.$$

Proof. See Appendix B.7. □

Financial hedging privileges currencies that minimize transaction costs. Increasing the foreign bid-ask spread always reduces the gains from denominating in a foreign currency $\frac{d\Delta_{ij}F}{d\Delta_{bid/ask}\tau^*} < 0$. Per Krugman (1980); Goldberg and Tille (2008), three types of pricing regimes occur:

1. Producer currency pricing, since $\Delta_{bid/ask}\tau_i = 1$

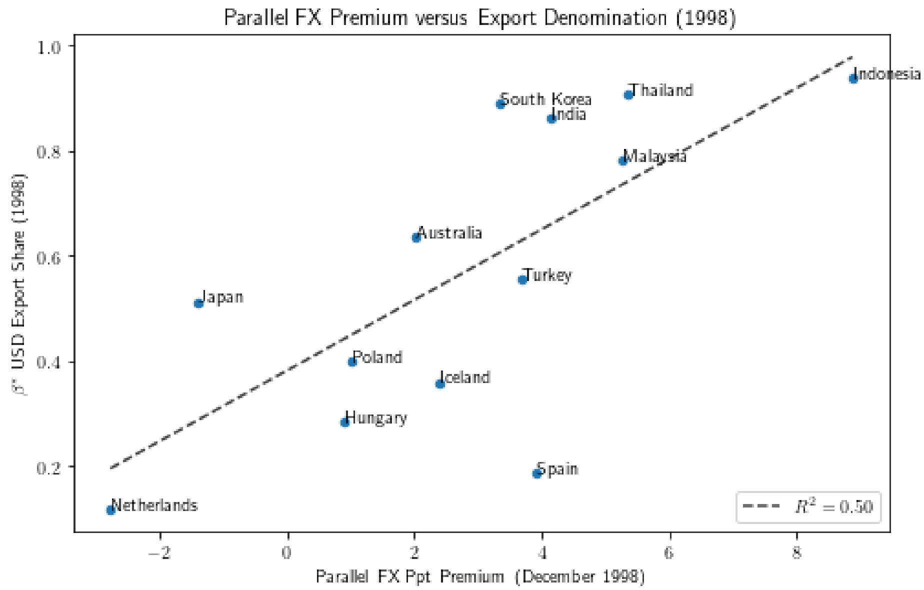

Figure 7: This scatter plot compares the cross-country currency denomination patterns versus each exporting country's parallel FX premium, measured as the percentage point spread in the Dollar exchange rate.

2. Local currency pricing, since $\Delta_{bid/ask}\tau_j^* = 1$

3. Dominant currency pricing, since $\Delta_{bid/ask}\tau_i^{\$} \downarrow 1$ and $\Delta_{bid/ask}\tau_j^{\$} \downarrow 1$.

PCP minimizes the seller's transaction costs; LCP minimizes the buyer's transaction costs; and DCP represents a second-best alternative for both agents.

Transaction costs are a prominent aspect of exchange rate regimes prior to the 21st century (Vries, 1987, 1996). The source of these transaction costs came in the form of parallel exchange rate markets. Although the government would maintain an "official" exchange rate, sanctioned for a particular economic activity or subject to quantity restrictions, a "parallel" exchange rate would develop through unofficial channels. These rates were at a premium depending on the relative supply and demand of the money in question.

Figure 7 is a cross-country scatterplot of the parallel FX premium relative to the USD export shares found in Boz et al. (2022). The parallel FX premium data are taken from Ilzetzki, Reinhart and Rogoff (2019), which was manually collected from the World Currency Yearbook. The year 1998 was chosen to maximize data availability, since the World Currency Yearbook was discontinued afterwards, while currency invoicing in trade generally increases in coverage across the years.

Consistent with the insights of this subsection, this figure shows a positive correlation between the FX premium and the USD export share. The FX premium is a transaction tax on the unofficial and official exchange rate of the market. Therefore, countries that have a larger FX premium, such as Indonesia, Thailand, and Malaysia, all have larger USD export shares. Although this is indirect evidence of the mechanism, it provides a concrete application relating currency choice to financial frictions.

B.4 Firm Liquidity

Firm liquidity also measures the value of financial hedging. Intuitively, the seller can use the trade contract to hedge against default. For example, if the seller is poorly capitalized, it may choose to price in the producer currency to avoid exposing itself to exchange rate movements. In contrast, if the seller is well-capitalized, it may instead price in the local currency to capture risk-sharing gains.

Following Froot, Scharfstein and Stein (1993), I microfound firm liquidity using a costly state verification setting. Costly state verification was developed by Townsend (1979) to rationalize debt as the optimal contract in the corporate finance environment. In this model, I find that the value of financial hedging is measured by the firm's probability of default and the distress price of exchange rates. This is consistent with the theoretical insights in Bruno and Shin (2015); Eren, Malamud and Zhou (2023).

There are now three agents in the model: the seller i, the buyer j and the lender ℓ. All agents are now risk neutral and discount time at rate 1.[4] Consequently, risk and time preferences are identical so that financial hedging occurs only endogenously. The seller i raises capital from the lender ℓ at time t equal to I. In return, the seller repays D to the lender ℓ in the next period. The seller can also default on its promise. If the seller defaults, the lender must engage in costly state verification paying a cost $c \geq 0$ to verify and seize the seller's profits. In this setting, it is known that optimal contracts are debt and equity. Thus, D is a constant.

The objective of the seller is to maximize

$$\max_{P_0, \beta^\intercal, I, D} \underbrace{\mathbb{E}\left[\pi\left(P, Q, I, x\right) - D\right]^+}_{\text{Expected Residual Profits}}.$$

The seller jointly chooses prices, currency denomination, quantities, investment, and the level of debt. When investing, the seller reduces future costs but increases the costs of default. When profits fall below the default threshold D, the seller must forfeit the profits to the lender and receive nothing.

The lender only participates if the net present value of the future debt obligation D

[4] Risk neutrality and absence of time discounting are invoked to simplify expressions.

exceeds the investment cost. This creates a standard investment constraint,

$$I \leq \mathbb{E}\left[\underbrace{D1_{\pi>D}}_{\text{No Default}} + \underbrace{(\pi - c)\,1_{\pi\leq D}}_{\text{Default}}\right].$$

The constraint states that the total capital offered by the lender ℓ is less than or equal to the expected payment. All other constraints—incentive, participation, and nominal rigidity—are kept constant in this analysis.

Proposition 10. *Define G as the CDF of the equilibrium profits and g as its corresponding PDF. Let D^* be the equilibrium debt level and define the "distress price" of exchange rates,*

$$\underbrace{s_{\pi\leq D}}_{\textit{Distress Price}} := \underbrace{(1 - G(D^*))\,\mathbb{E}[s \mid \pi^* = D^*]}_{\textit{FX Conditional at Default Threshold}} + \underbrace{G(D^*)\,\mathbb{E}[s \mid \pi^* < D^*]}_{\textit{FX Conditional in Default}}.$$

A sufficient statistic for the relative cost of FX hedging is given by

$$\Delta_{ij}F := \underbrace{\frac{cg(D^*)}{1 - G(D^*) - cg(D^*)}}_{\textit{Default Intensity}} \times \underbrace{s_{\pi\leq D}}_{\textit{Distress Price}}.$$

Proof. See Appendix B.8. □

The firm is unable to financially hedge due to costly state verification. Consequently, the cost of hedging measures the cost of exchange rate volatility, which is captured by the intensity of the default and the price of the distress. The default intensity is a scalar which captures how much the lender up-charges the cost of capital due to default. Consequently, it depends on both the cost of verification $c \geq 0$ and the intensity of the default at the equilibrium level of debt $g(D^*) \geq 0$.

The distress price reflects whether the currency appreciates during firm default. Unlike the default intensity term, $s_{\pi\leq D}$ determines whether each pair of currencies is likely to appreciate or depreciate. For example, the seller is likely to prioritize the producer currency, since these remain flat during global business cycle downturns. However, the intensity of this incentive is regulated by the cost of default $cg(D^*)$.

Empirically, this suggests that in transaction-level pricing data, a larger seller should price more in the local currency because they can easily absorb financial risk. Both Amiti, Itskhoki and Konings (2022) and Devereux, Dong and Tomlin (2017) find evidence of a significant connection between size and currency choice, even after controlling for theories of real hedging. In the data, larger Belgium firms are more likely to export in the local currency to extra-EU countries than smaller firms, within time, destination, and controlling for competitor pricing decisions. These firms share risk efficiently and use the local currency pricing decision to extract price concessions from buyers, which may be small retailers or households.

B.5 Proof of Proposition 7

To determine the Euler equation wedges, we now use the Arrow-Pratt portfolio choice method. Let the buyer's next period utility be summarized by the indirect utility function $U(W, x)$. The optimal wealth share of currencies today $\theta \in \mathbb{R}^n$ must satisfy

$$\max_{\theta} \mathbb{E}\left[U(W, x)\right]$$
$$\text{s.t. } W \leq W_0\left[R + \theta^{\intercal}\left(R^* S - R\right)\right]$$

It is well-known from the Arrow-Pratt solution technique that $\theta^{\intercal} = RRA_{ss}^{-1}\Sigma^{-1}\mathbb{E}\left[R^* S - R\right]$ (Pratt, 1964; Arrow, 1971). However, as an exercise of validating Lemma 2, I prove it with the projection technique.

Begin by assuming the technical conditions hold (x and θ are real, continuous, and bounded). Denote q as a set of hypothetical Arrow-Debreu security prices which span the filtration $\mathcal{F}^x$. The trading strategy of purchasing an Arrow-Debreu security yields the rate of return $1(x) - Rq_x$ where $1(x)$ is the step function. With the fully indexed solution

$$W \leq W_0\left[R + \zeta^{\intercal}\left(1(x) - Rq_x\right)\right]$$

the first-order conditions are given by the point-wise condition

$$U' W_0 \left(1(x) - Rq_x\right)^{\intercal} \mu(x) = 0.$$

Taking $x \to \mathbb{E}[x]$ and linearize this condition. Note that as a consequence $1(x) \to Rq_x$. In this limit,

$$\bar{U}'' W_0^2 \zeta^{\intercal}\left(1(x) - Rq_x\right)\left(1(x) - Rq_x\right)^{\intercal} + \bar{U}' W_0 \left(1(x) - Rq_x\right)^{\intercal} = 0.$$

rearrange this as

$$\zeta^{\intercal} = -\frac{\bar{U}'}{\bar{U}'' \cdot W_0}\left[\left(1(x) - Rq_x\right)\left(1(x) - Rq_x\right)^{\intercal}\right]^{-1}\left(1(x) - Rq_x\right)^{\intercal}.$$

Consequently, the buyer's optimal wealth in each state is

$$W = W_0 \left[R + \zeta^{\intercal} \left(1\left(x\right) - Rq_x \right) \right].$$

In the exchange-rate indexed solution, the buyer's optimal wealth in each state is instead

$$W = W_0 \left[R + \theta^{\intercal} \left(R^* S - R \right) \right].$$

Thus, the projection method implies that

$$\begin{aligned} \theta^{\intercal} &\approx \text{Var}\left(R^*S - R\right)^{-1} \text{Cov}\left(R^*S - R, 1_x\right) \zeta^{\intercal} \\ &= R * RRA^{-1} \text{Var}\left(R^*S - R\right)^{-1} \text{Cov}\left(R^*S - R, 1\left(x\right)\right) \\ &\quad \times \mathbb{E}\left[\left(1\left(x\right) - Rq_x\right)\left(1\left(x\right) - Rq_x\right)^{\intercal}\right]^{-1} \mathbb{E}\left(1\left(x\right) - Rq_x\right)^{\intercal}. \end{aligned}$$

Note that the returns on the trading strategy can be rewritten as a multilinear mapping of the Arrow-Debreu securities, by virtue of the AD securities spanning the filtration. Consequently, the linear projection recovers the expected return of the strategy

$$\begin{aligned} &\text{Cov}\left(R^*S - R, 1_x\right) \mathbb{E}\left[\left(1\left(x\right) - Rq_x\right)\left(1\left(x\right) - Rq_x\right)^{\intercal}\right]^{-1} \mathbb{E}\left(1\left(x\right) - Rq_x\right)^{\intercal} \\ =&\mathbb{E}\left[R^*S - R\right] \end{aligned}$$

as $x \to \mathbb{E}[x]$. Thus recovering the Arrow-Pratt portfolio solution $\theta^{\intercal} = RRA^{-1}\Sigma\mathbb{E}\left[R^*S - R\right]$ as intended.

Now, to finalize the proof, recall that the buyer's first order condition states that

$$\mathbb{E}\left[U' W_0 \left(R^*S - R\right)\right] = 0.$$

Using $U'W_0$ as M^j it follows that $\mathbb{E}\left[\frac{M^j}{\mathbb{E}[M^j]} R^*S\right] = R$. Meanwhile, because the seller is risk neutral we know that $\mathbb{E}\left[R^*S\right] = \mathbb{E}\left[\frac{M^i}{\mathbb{E}[M^i]} R^*S\right]$. Combining, we get

$$\begin{aligned} \mathbb{E}\left[R^*S - R\right] &= \mathbb{E}\left[\left(\frac{M^i}{\mathbb{E}\left[M^i\right]} - \frac{M^j}{\mathbb{E}\left[M^j\right]}\right) R^*S\right] \\ &= R\Delta_{ij}F. \end{aligned}$$

Rewriting, $\theta^{\intercal} = RRA^{-1}\Sigma^{-1}R\Delta_{ij}F$ so that $\Delta_{ij}F := R^{-1} \cdot RRA \cdot \Sigma \cdot \theta$.

B.6 Proof of Proposition 8

The firm now jointly maximizes

$$\max_{P_0,\beta^{\intercal},B,I} \mathbb{E}\left[M^i\left(\underbrace{\pi\left(P,Q,I,x\right)}_{\text{Gross Income}} - \underbrace{R^*\left(B^*\right)SB^*}_{\text{Foreign CC}} - \underbrace{R\left(B\right)B}_{\text{Domestic CC}}\right)\right]$$

such that

$$B + B^* \geq I$$

give the same buyer IC, buyer PC, and nominal rigidity constraints. Rewrite the problem in terms of the risk neutral measure. The first order conditions of the problem are identical for the price, denomination, and quantity margins. It now features the bond margin satisfying

$$\mathbb{E}\left[M^i\left(\pi_I - \partial_{B^*}R^*SB - R^*S\right)\right] = 0$$
$$\mathbb{E}\left[M^i\left(\pi_I - \partial_B RB - R\right)\right] = 0.$$

Differencing the two conditions, we get

$$\mathbb{E}\left[M^i\left(\left(1+\partial_{b^*}r^*\right)R^*S - \left(1+\partial_b r\right)R\right)\right] = 0 \Rightarrow \frac{1+\tau^{i*}}{1+\tau^i} = \frac{1+\partial_b r}{1+\partial_{b^*}r^*}$$

using the fact that the seller has competitive preferences, it follows that

$$\frac{1+\tau^{j*}}{1+\tau^j}\frac{R}{R^*}S = \mathbb{E}\left[\frac{M^j}{\mathbb{E}\left[M^j\right]}S\right] = F.$$

Rearranging this equation

$$\frac{1+\tau^{j*}}{1+\tau^j} = F\frac{R^*}{R}$$

and combining with earlier

$$\Delta_{ij}F := \frac{1+\partial_b r}{1+\partial_{b^*}r^*} - F\frac{R^*}{R}.$$

The rest of the problem remains identical since the investment margin is linearly separable.

B.7 Proof of Proposition 9

Recall that the seller receives exchange rate risk and the buyer sells exchange rate risk. Consequently, the seller and buyer equate their Euler equations to the realized bid and ask price of S. This corresponds to the following two equations,

$$1=\underbrace{\mathbb{E}\left[M^i\frac{S_{ic}^{bid'}}{S_{ic}^{ask}}R_c\right];}_{\text{Seller Long 1 FX}}\qquad \underbrace{1=\mathbb{E}\left[M^j\frac{S_{jc}^{ask'}}{S_{jc}^{bid}}R_c\right]}_{\text{Buyer Short 1 FX}}.$$

To derive the wedges τ^i and τ^j, we need to rearrange the definitions in terms of the hypothetical market rate.

$$1=\mathbb{E}\left[M^iS_c\frac{1+\tau_{ic}^{bid}}{1+\tau_{ic}^{ask}}R_c\right]=\mathbb{E}\left[M^jS_c\frac{1+\tau_{jc}^{ask}}{1+\tau_{jc}^{bid}}R_c\right]$$

Since t-costs are constant across time

$$1+\tau^{i*}=\frac{1+\tau_i^{ask*}}{1+\tau_i^{bid*}};\qquad 1+\tau^{j*}=\frac{1+\tau_j^{bid*}}{1+\tau_j^{ask*}}$$

so

$$\Delta_{ij}F=\frac{\Delta_{bid/ask}\tau_i}{\Delta_{bid/ask}\tau_i^*}-\frac{\Delta_{bid/ask}\tau_j^*}{\Delta_{bid/ask}\tau_j}.$$

B.8 Proof of Proposition 10

The seller's objective is given by

$$\max_{p_0,\beta,I,D}\int_X\left(\pi\left(P,Q,I,x\right)-D\right)1_{\pi\geq D}d\mu$$

subject to the constraints

$$\int_X\left[\pi\left(P,Q,I,\omega\right)-c\right]1_{\pi<D}d\mu+\int_X D1_{\pi\geq D}d\mu\geq I$$
$$\left(1-\delta\right)V_{(x)}^{-1}\left(P\right)+\delta\bar{V}^{-1}\left(\mathbb{E}\left[M^jP\right]\right)=Q$$

to solve this problem, we characterize the Lagrangian but instead differentiate on the CDF functions, which are implicit functions of the choice of prices and quantities.[5]

Assume the CDF for x is given by F and is continuously differentiable with pdf f. Define the CDF of the equilibrium profits as $G\left(\rho\right)$. Since π is continuously differentiable, the pdf of profits $g\left(\rho\right)$ is well defined. Since $\pi\left(P\left(x\right),Q\left(x\right),x\right)$ is a function of the choice variable $P\left(x\right)$ and $Q\left(x\right)$, we can specify $\hat{G}\left(\rho;P\left(x\right),Q\left(x\right)\right)$ as the CDF taking into account the choice of price and quantity. Assume regularity conditions on the distribution $\lim_{L\to\infty}Lg\left(L\right)=0$ and $\lim_{L\to-\infty}G\left(L\right)=0$.

[5]Special thanks to Sebastian Bauer for help on this proof.

The maximization is equivalent to characterizing the Lagrangian,

$$\begin{aligned}
\mathcal{L} = & \min_{\eta(x_1),\eta,\lambda\geq 0} \max_{p_0,\beta^\intercal,q(\omega),D,I} \int_D^\infty (\rho - D)\, d\hat{G}(\rho; P(x), Q(x)) \\
& + \lambda \left(\int_\infty^D [\rho - c]\, d\hat{G}(\rho; P(x), Q(x)) + \int_D^\infty D d\hat{G}(\rho; P(x), Q(x)) - I \right) \\
& + \int \eta(x) \left[(1-\delta) V_{(x)}^{-1}(P) + \delta \bar{V}^{-1}(\mathbb{E}[P]) - Q(x) \right] dF(x)
\end{aligned}$$

Specifically, the CDF $\hat{G}$ is defined as

$$\begin{aligned}
\hat{G}(\rho; P(x), Q(x)) &= \mathbb{E}\left[1_{\pi(P(x),Q(x),x)\leq\rho}\right] \\
&= \mathbb{E}\left[1_{\pi^* + \int_{Q^*(x)}^{Q(x)} \int_{P^*(x)}^{P(x)} \partial_{PQ}\pi(P,Q,x) dPdQ \leq \rho}\right] \\
&= \int 1_{\pi^*(x)\leq\rho - \int_{Q^*(x)}^{Q(x)} \int_{P^*(x)}^{P(x)} \partial_{PQ}\pi(P,Q,x) dPdQ} dF(x) \\
&= \int G\left(\rho - \int_{Q^*(x)}^{Q(x)} \int_{P^*(x)}^{P(x)} \partial_{PQ}\pi(P, Q, x)\, dPdQ \mid x \right) dF(x)
\end{aligned}$$

thus differentiating

$$\begin{aligned}
\partial_{Q(x)} d\hat{G}(\rho; P^*(x), Q^*(x)) &= -\int g'(\rho \mid x)\, \partial_Q \pi(P^*, Q^*, x)\, 1_x dF(x) \\
&= -g'(\rho \mid x)\, \partial_Q \pi(P^*, Q^*, x)\, dF(x)
\end{aligned}$$

moreover, differentiating by p_0

$$\begin{aligned}
\partial_{P_0} d\hat{G}(\rho; P^*(x), Q^*(x)) &= -\int g'(\rho \mid x)\, \partial_P \pi(P^*, Q^*, x)\, dF(x) \\
&= -\mathbb{E}\left[g'(\rho \mid x)\, \partial_P \pi(P^*, Q^*, x)\right]
\end{aligned}$$

and

$$\partial_\beta d\hat{G}(\rho; P^*(x), Q^*(x)) = -\mathbb{E}\left[g'(\rho \mid x)\, \partial_P \pi(P^*, Q^*, x)\, s\right].$$

Using this result, we can now characterize the first order conditions as if they were directly differentiating on the CDF. Differentiating over p_0

$$\begin{aligned}
0 = & -\int_D^\infty \int_X (\rho - D)\, g'(\rho \mid x)\, \partial_P \pi^*(x)\, dF(x)\, d\rho \\
& - \lambda \left(\int_{-\infty}^D \int_X [\rho - c]\, g'(\rho \mid x)\, \partial_P \pi^*(x)\, dF(x)\, d\rho + \int_D^\infty \int_X D g'(\rho \mid x)\, \partial_P \pi^*(x)\, dF(x)\, d\rho \right) \\
& + \int_X \eta(x) \left[(1-\delta)\, \partial_P V_{(x)}^{-1}(P) + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}(\mathbb{E}[P]) \right] dF(x)
\end{aligned}$$

since our functions are bounded, we can apply Fubini's Theorem and tuck the integral over D inside. Integrating by parts,

$$\begin{aligned}
\int_D^\infty (\rho - D)\, g'(\rho \mid x)\, d\rho &= \lim_{L\to\infty} Lg(L \mid x) - Dg(D \mid x) - \int_D^\infty g(\rho \mid x)\, d\rho - \int_D^\infty Dg'(\rho \mid x)\, d\rho \\
&= -Dg(D \mid x) - (G(\infty \mid x) - G(D \mid x)) - D\left[g(\infty \mid x) - g(D \mid x)\right] \\
&= -(1 - G(D \mid x))
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^D [\rho - c]\, g'(\rho \mid x)\, d\rho &= (D - c)\, g(D \mid x) - G(D \mid x) \\
\int_D^\infty Dg'(\rho \mid x)\, d\rho &= D\left(g(\infty \mid x) - g(D \mid x)\right) = -Dg(D \mid x)
\end{aligned}$$

Using these conditions we get

$$\begin{aligned}
0 = &\int_X [1 - G(D \mid x)]\, \partial_P \pi^*(x)\, dF(x) \\
&+ \lambda\left(\int_X [G(D \mid x) - (D - c)\, g(D \mid x)]\, \partial_P \pi^* dF(x) + \int_X Dg(D \mid x)\, \partial_P \pi^* dF(x)\right) \\
&+ \int_X \eta(x)\left[(1 - \delta)\, \partial_P V_{(x)}^{-1}(P) + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}(\mathbb{E}[P])\right] dF(x) \\
= &\int_X [1 - (1 - \lambda)\, G(D \mid x) + cg(D \mid x)]\, \partial_P \pi^*(x)\, dF(x) \\
&+ \int_X \eta(x)\left[(1 - \delta)\, \partial_P V_{(x)}^{-1}(P) + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}(\mathbb{E}[P])\right] dF(x)
\end{aligned}$$

similarly for β we get the first order condition

$$\begin{aligned}
0 = &-\int_D^\infty \int_X \rho g'(\rho \mid x)\, \partial_P \pi^*(x)\, s dF(x)\, d\rho \\
&- \lambda \int_0^D \int_X [\rho - c]\, g'(\rho \mid x)\, \partial_P \pi^*(x)\, s dF(x)\, d\rho \\
&- \lambda \int_D^\infty \int_X Dg'(\rho \mid x)\, \partial_P \pi^*(x)\, s dF(x)\, d\rho \\
&+ \int_X \eta(x)\left[(1 - \delta)\, \partial_P V_{(x)}^{-1}(P)\, s + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}(\mathbb{E}[P])\, \mathbb{E}[s]\right] dF(x)
\end{aligned}$$

since the s term does not change the integration identities as they hold x fixed, we get

$$\begin{aligned}
0 = &\int_X [1 - (1 - \lambda)\, G(D \mid x) + cg(D \mid x)]\, \partial_P \pi^*(x)\, s dF(x) \\
&+ \int_X \eta(x)\left[(1 - \delta)\, \partial_P V_{(x)}^{-1}(P)\, s + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}(\mathbb{E}[P])\, \mathbb{E}[s]\right] dF(x).
\end{aligned}$$

For the quantity margin, our first order condition is

$$
\begin{aligned}
0 = &- \int_D^\infty \rho g' \left(\rho \mid x\right) \partial_{Q(x)} \pi^* \left(x\right) dF \left(x\right) d\rho \\
&- \lambda \left(\int_0^D \left[\rho - c\right] g' \left(\rho \mid x\right) \partial_{Q(x)} \pi^* \left(x\right) dF \left(x\right) d\rho + \int_D^\infty D g' \left(\rho \mid x\right) \partial_{Q(x)} \pi^* \left(x\right) dF \left(x\right) d\rho \right) \\
&- \int_X \eta \left(x\right) 1_x dF \left(x\right)
\end{aligned}
$$

once again using the integration identities and dividing through by $\int 1_x dF(x)$, we get:

$$
0 = \left[1 - \left(1 - \lambda\right) G \left(D \mid x\right) + \lambda c g \left(D \mid x\right)\right] \partial_{Q(x)} \pi^* \left(x\right) - \eta \left(x\right)
$$

define $\frac{dF^i(x)}{dF} = 1 - (1 - \lambda^*) G(D^* \mid x) + \lambda^* c g (D^* \mid x)$ and $\frac{dF^j}{dF} = 1$ as the constant in front of the revenue choice thus rewriting

$$
\begin{aligned}
\mathbb{E}^{\mu^i} \left[\partial_P \pi^*\right] + \mathbb{E} \left[\eta \left(x\right) \left\{ \left(1 - \delta\right) \partial_P V^{-1} + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1} \right\} \right] &= 0 \\
\mathbb{E}^{\mu^i} \left[\partial_P \pi^* s\right] + \mathbb{E} \left[\eta \left(x\right) \left\{ \left(1 - \delta\right) \partial_P V^{-1} s + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1} \mathbb{E} \left[s\right] \right\} \right] &= 0 \\
\frac{dF^i}{dF} \partial_{Q(x)} \pi^* - \eta \left(x\right) &= 0
\end{aligned}
$$

This is equivalent to the first-order conditions of the full problem where $\frac{M^i / \mathbb{E}\left[M^i\right]}{M^j / \mathbb{E}\left[M^j\right]} = \frac{dF^i}{dF}$ and $M^j := 1$. Intuitively, $1 - (1 - \lambda^*) G (D^* \mid x)$ captures the direct effect of changing profits when the firm is in the interior of the default region. The term $\lambda^* c g (D^* \mid x)$ captures the marginal effect of moving the profits nearer the no default region. When the profits marginally enter the no-default region, they reduce the probability that the investor needs to pay c.

To pin down $\frac{dF^i(x)}{dF}$ we now use the first order condition on the debt choice to recover λ.

$$
\begin{aligned}
0 = - \int_D^\infty dG \left(\rho\right) + \lambda \left[D - c\right] g \left(D\right) - \lambda D g \left(D\right) + \lambda \int_D^\infty dG \left(\rho\right) \\
\left(1 - G \left(D\right)\right) = \lambda \left(1 - G \left(D\right) - \left(1 - \delta\right) D g \left(D\right)\right)
\end{aligned}
$$

thus

$$
\lambda^* = \frac{1 - G \left(D^*\right)}{1 - G \left(D^*\right) - c g \left(D^*\right)} = 1 + \frac{c g \left(D^*\right)}{1 - G \left(D^*\right) - c g \left(D^*\right)}
$$

thus

$$
\begin{aligned}
&1-(1-\lambda)\,G\left(D^{*}\mid x\right)+\lambda cg\left(D^{*}\mid x\right)\\
=&1+\frac{\left(1-G\left(D^{*}\right)\right)cg\left(D^{*}\mid x\right)+cg\left(D\right)G\left(D\mid x\right)}{1-G\left(D^{*}\right)-cg\left(D^{*}\right)}
\end{aligned}
$$

Due to profits being entirely characterized by x, we have the identity $G\left(D\mid x\right)=1_{\pi(x)\leq D}$. Consequently, $g\left(D^{*}\mid x\right)$ is formalized using the Dirac delta function at D^{*} . From the risk neutrality of the buyer, it follows that

$$
\begin{aligned}
\mathbb{E}\left[\frac{M^{i}}{\mathbb{E}\left[M^{i}\right]}s\right]-\mathbb{E}\left[\frac{M^{j}}{\mathbb{E}\left[M^{j}\right]}s\right]&=\mathbb{E}\left[\left(\frac{\left(1-G\left(D^{*}\right)\right)cg\left(D^{*}\mid x\right)+cg\left(D\right)G\left(D\mid x\right)}{1-G\left(D^{*}\right)-cg\left(D^{*}\right)}\right)s\right]\\
&=\frac{cg\left(D^{*}\right)}{1-G\left(D^{*}\right)-cg\left(D^{*}\right)}\mathbb{E}\left[1_{\pi^{*}\leq D^{*}}s\right]\\
&\quad+\frac{1-G\left(D^{*}\right)}{1-G\left(D^{*}\right)-cg\left(D^{*}\right)}c\int_{\Omega}g\left(D^{*}\mid x\left(\omega\right)\right)s\left(\omega\right)d\mu\left(\omega\right)
\end{aligned}
$$

Integrating over the Dirac delta measure and using a change of measure,

$$
\begin{aligned}
\int_{\Omega}g\left(D^{*}\mid x\left(\omega\right)\right)s\left(\omega\right)d\mu\left(\omega\right)&=\int_{-\infty}^{\infty}\int_{X}g\left(D^{*}\mid\rho\right)sdF\left(x\mid\rho\right)dG\left(\rho\right)\\
&=\mathbb{E}\left[s\mid\pi^{*}=D^{*}\right]g\left(D^{*}\right)
\end{aligned}
$$

This gives us

$$
\begin{aligned}
\Delta_{ij}F&=\frac{\left(1-G\left(D^{*}\right)\right)\mathbb{E}\left[s\mid\pi^{*}=D^{*}\right]+\mathbb{E}\left[1_{\pi^{*}\leq D^{*}}s\right]}{1-G\left(D^{*}\right)-cg\left(D^{*}\right)}cg\left(D^{*}\right)\\
&=\frac{cg\left(D^{*}\right)s_{\pi\leq D}}{1-G\left(D^{*}\right)-cg\left(D^{*}\right)}
\end{aligned}
$$

where $s_{\pi\leq D}=\left(1-G\left(D^{*}\right)\right)\mathbb{E}\left[s\mid\pi^{*}=D^{*}\right]+G\left(D^{*}\right)\mathbb{E}\left[s\mid\pi^{*}\leq D^{*}\right]$ is the distress price.

C Additional Empirical Tests on Trade Invoicing

C.1 Cross-Sectional Tests of Financial Hedging

A key prediction of the theory is that producers that FX hedge are more likely to invoice in the foreign currency if quantities are sticky. By having access to FX hedging markets, an exporter is better able to bear the foreign exchange rate risk associated with foreign currency of invoicing. Crucially, I find that this effect matters only through the interaction

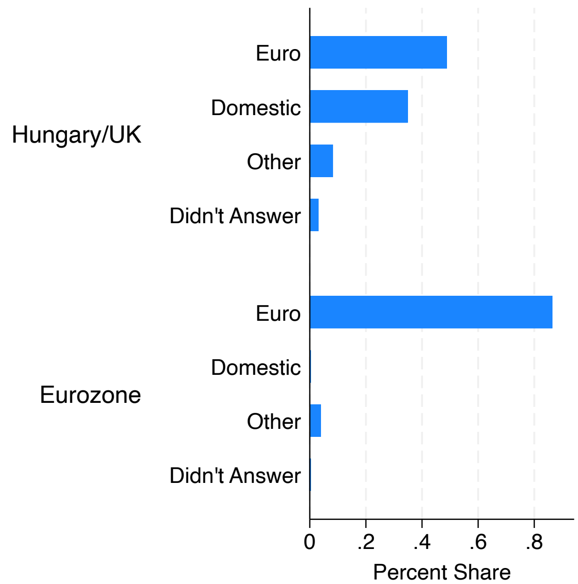

Figure 8: Equal weighted tabulation of response rates to "in which currency do you set your prices in foreign countries?" Eurozone consists of Austria, Germany, Spain, France, and Italy.

of the quantity margin. The producer does not care about risk sharing if quantities are flexible, such as direct sale to retail markets.

The null hypothesis is that the cost of financial hedging is irrelevant in trade invoicing. To test my model, I would ideally use exogenous variation in either quantity stickiness or access to FX hedging markets. Due to data limitations, I will show survey-based evidence consistent with the theory. I will use the European Firms in a Global Economy (EFIGE) survey by Bruegel, which was conducted by Altomonte and Aquilante (2012). This data set has been analyzed by Lyonnet, Martin and Mejean (2022) in a similar context, except that it has not focused on the explicit interaction with the quantity dimension, which is novel in my theory of invoicing.

The survey consists of 15,000 companies with more than ten employees. It covers Austria, Germany, Spain, France, the UK, Hungary, and Italy and is concentrated in the manufacturing sector. It is a cross section of questionnaires based on 2008 balance sheet data, sampled to be representative with a sector-country pair. The large questionnaire covers many aspects of the firm's pricing behavior and international activities. Although the questionnaire is anonymized at the firm level, it also includes Amadeus-linked balance sheet data.

The data contains a proxy on trade invoicing. It asks survey participants, "In which currency do you set your prices in foreign countries?" A tabulation of the response rates is shown in Figure 8. Combined with the country of incorporation, it is possible to determine a binary PCP outcome. For example, a French firm that reports "Euro" or "Domestic" is coded

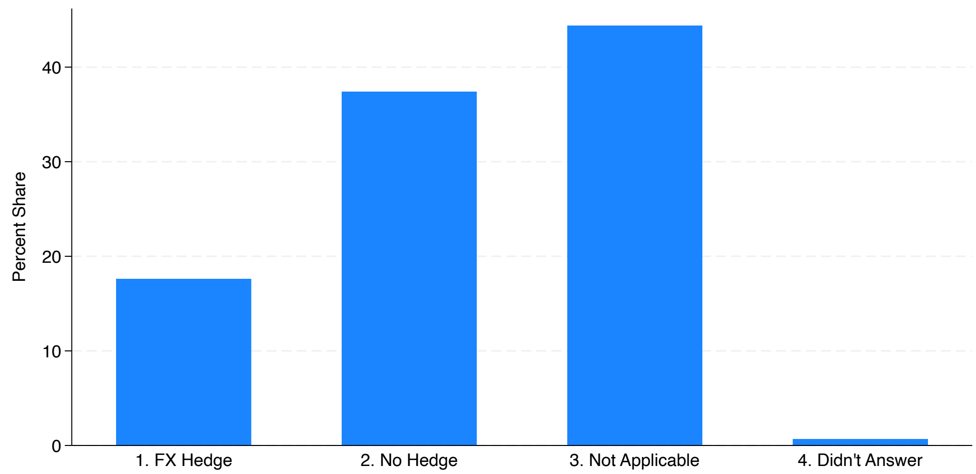

Figure 9: Equal-weighted response rates to the question "How do you deal with exchange rate risk? Which of the following statements is similar to what your firm does?

as a producer that invoices in PCP. A caveat with this measurement is that it is possible that firms sell in producer and foreign currencies for some exports. In this scenario, my measurement of the PCP binary outcome will suffer from classic measurement error.

In addition, the survey also measures whether or not a firm hedges their foreign exchange rate risk. In particular, the survey asks: "How do you deal with the exchange rate risk? Which of the following statements is similar to what your firm does?" In response, the survey participant can choose from one of the following answers:

1. "I use a foreign exchange risk protection"
2. "I do not normally hedge against exchange rate risk"
3. "The question is not applicable, as I only sell to countries with the same currency of my domestic market"
4. "Didn't Answer/Didn't Know"

I interpret the first response as a sign that a firm uses financial FX hedges, often in the form of forward contracts or swaps.

In Figure 9, I plot the response rates for each of these answers. The number corresponds to the enumerated list of responses. There are a sizable share of firms that do not find the question applicable because they are not exposed to foreign exchange rate risk. However, of the firms that actively choose to hedge or "do not normally hedge against exchange rate risk," around 1/3 of firms use foreign exchange rate risk protection. This is an imperfect measure of access to financial markets. Under the assumptions of Modigliani-Miller, financial hedging

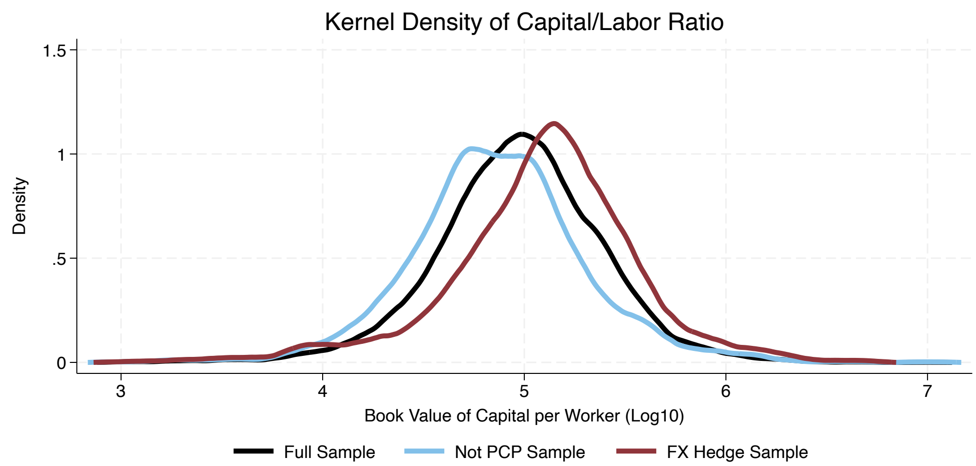

Figure 10: Equal-weighted kernel density plot of the capital intensity for each producer. Balance sheet data is directly linked in the EFIGE survey data.

does not create firm value. Consequently, it is possible that firms that do not hedge can hedge and choose not to. By itself, the answer to this question does not directly test the theory of financial hedging.

Finally, to test my theory, I need an empirical proxy for the degree of quantity stickiness. Theorem 1 shows that the invoicing choice is proportional to

$$\text{PCP} \propto \underbrace{\frac{\delta \partial_P \bar{\pi}}{\bar{\pi}_{PP}} \times \frac{\Delta_{ij} F}{\text{Var}\,(s)}}_{\text{Financial Hedging}} + \underbrace{\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} b_{xs}}_{\text{Real Hedging}} . \tag{30}$$

In this formula δ is a measure of the share of fixed quantities in trade and $\Delta_{ij}F$ is the relative cost of FX risk sharing. Therefore, the central prediction of my theory is that firms that are more exposed to the financial risks of a currency care more about the relative cost of FX hedging.

I proxy for sticky quantities $\frac{\delta \partial_P \bar{\pi}}{\bar{\pi}_{PP}}$ with the capital intensity of the firm's production function. This has been used in the New Keynesian literature as a measure of real rigidities in production (Eichenbaum and Fisher, 2007). It serves as a helpful proxy because it captures adjustment costs. For example, a wine producer cannot flexibly adjust quantities because of the time-to-build nature of their good. This makes wine production capital-intensive, as an example. To measure capital intensity, I use firm-level data on the capital-labor ratio, measured as the book value of tangible capital stock per employee.

Figure 10 plots the empirical distribution of the capital intensity across exporters. In the full sample, the capital intensity tends to be log-normally distributed. It appears that the local currency pricing sample tends to be less capital intensive, while the sample of firms that FX hedge tends to be more capital intensive. This is broadly consistent with a story of selection into hedging: firms FX hedge because their production is more subject to financial risk, due to its capital intensity. Consequently, it is unlikely that the FX hedging variable is exogenously determined in the sample.

To test my model, I lean on the variation in the capital intensity of production to "shift" the relevance of financial hedging. In particular, I estimate the linear probability model:

$$\text{PCP}_i = \gamma_1 \log\left(K/L\right)_i + \gamma_2 1\left\{\text{FX Hedge}_i\right\} \tag{31}$$

$$+ \gamma_3 \log\left(K/L\right)_i \times 1\left\{\text{FX Hedge}_i\right\} + X_i + \text{FEs} + \epsilon_i \tag{32}$$

In this equation, I have colored the capital intensity and the FX hedge indicator variable to relate to Equation (30). Capital intensity captures quantity stickiness and the FX hedge indicator measures the gains from FX risk sharing. The model is consistent with the theory if the coefficient $\gamma_3 < 0$ is negative. In this equation X_i controls for real hedging, i.e. foreign input shares, the number of foreign affiliates, and export shares to different regions. The fixed effects control is an interacted bin of industry, country, size, age, and turnover. The size, age, and turnover bins are coarsely defined over several bins. Industry is a SIC 2 code within the manufacturing sector. Finally, I keep only firms that specify how they set prices and are not subject to pricing regulation.

	(1)	(2)	(3)	(4)
γ_2 (FX Hedge Indicator)		-0.011	-0.020	0.438^{***}
		(0.014)	(0.014)	(0.154)
γ_1 (Capital-Labor Ratio)		0.014	0.016	0.033^{**}
		(0.015)	(0.014)	(0.015)
γ_3 (Interaction)				-0.091^{***}
				(0.030)
Obs.	7610	7610	7610	7610
Adjusted R^2	0.24	0.24	0.33	0.33
Sets Price Sample	Yes	Yes	Yes	Yes
Country-Size-Sector-Age-Turnover FEs	Yes	Yes	Yes	Yes
Real Hedging Controls	No	No	Yes	Yes

In the regression table, I report the estimated coefficients across four specifications. The preferred specification is in the final column. In the first column, I consider only the variation that is explained by the fixed effects. In the second column and third column, I add the non-interacted variables and the controls. I find that the real hedging controls tend to be important in explaining variation in the currency of invoicing choice. However, the non-interacted effect of FX hedging and the capital labor ratio are insignificant. This is consistent with the theoretical predictions of my model, which show that by themselves, these variables are not meaningful in predicting the firm's trade invoicing behavior.

Instead, in the final column, I find the theoretically consistent prediction that the interaction is negative and statistically significant. It suggests that changing from the 5 to 95 percentile of capital intensity increases the probability of foreign currency invoicing by 12 ppt (for an FX hedged firm). Also, the non-interacted effects become important. I interpret this as saying that in the third column, the direct effects were insignificant because they conflated whether or not the firm was using the trade invoicing decision to share FX risk. This regression uses variation in the capital intensity of exporters among firms that FX hedge to estimate the importance of financial hedging. The results are meaningfully different from Lyonnet, Martin and Mejean (2022) in that they emphasize the quantity interaction with FX hedging. Additionally, the estimation imposes stronger controls and additional fixed effects to purge the real hedging explanation.

The findings do not depend on the proxy for quantity stickiness. Another measure of real rigidities in trade is size. The reason is because large firms tend to charge variable markups, which reduces the passthrough of prices into quantities. Indeed, many formulations of the New Keynesian model have this feature built in. In the table below, I run the additional specifications using this alternative measure for quantity stickiness: Across the board, we see

	(1)	(2)	(3)
γ_3 (Capital Intensity Interaction)	-0.091^{***}		
	(0.030)		
$\hat{\gamma}_3$ (Sales Interaction)		-0.134^{***}	
		(0.024)	
$\hat{\gamma}_3$ (Size Interaction)			-0.001^{***}
			(0.000)
Obs.	7610	7610	7610
Adjusted R^2	0.33	0.33	0.33
Sets Price Sample	Yes	Yes	Yes
Country-Size-Sector-Age-Turnover FEs	Yes	Yes	Yes
Real Hedging Controls	Yes	Yes	Yes

that the effect is negative. In the size and sales interaction, I code the size and sales bin in which each firm belongs as the variable. The survey does not directly report the total sales or the exact number of employees in each firm. The size category is broken up into buckets 10-20, 20-50, 50-250, >250 employees. Sales are also grouped into seven categories ranging from 1 million to 250 million in local currency.

These cross-sectional tests suggest meaningful variation between the types of firms that use trade invoicing to share the risk of FX. It is easy to imagine that sectors that have capital intensive output will tend to invoice in the currency that mitigates financial risk. It also suggests that contracts with enforcement, such as trade contracts between two businesses, will use trade invoicing to hedge financial risk. However, retailers are less likely to use trade invoicing to hedge financial risk because households do not commit to buying foreign goods.

C.2 Details on UK Trade Export Invoicing

On 23 June 2016, the United Kingdom voted in a referendum to leave the European Union. The vote resulted in elevated exchange rate volatility, changes in bilateral tariff rates with the EU, and a change in the policy commitments of the United Kingdom. "Brexit" also marked an aggregate shift away from producer currency pricing (pound) to dominant currency pricing (dollar) in export invoicing. From 2016 to 2022, the share of dollar invoicing rose from 30 percent to 45 percent, excluding exports to the US and EU. Similarly, the share of pound invoicing dropped from roughly 60 percent to less than 40 percent. In this section, I show how this transition from PCP to DCP was a response to the financial hedging properties of the USDGBP exchange rate.

Garofalo, Rosso and Vicquéry (2024) documents that firms more financially exposed were more likely to change from pound to dollar invoicing. This suggests that the transition was induced by the financial properties of the dollar exchange rate. In their paper, they measure each firm's financial exposure by total exports less imports in the pound, normalized by gross trade. In the Armington trade model, this measure is precisely

$$\frac{\text{Exports} - \text{Imports}}{\text{Total Trade}} = \frac{(1-\beta)\,PQ - (1-\gamma)CQ}{PQ} = 1 - \beta - \frac{1-\gamma}{\mu}.$$

In this equation, β is the foreign currency invoicing share, γ is the foreign currency import share, and the price $P = \mu C$ is the optimal markup over marginal costs C. Take the simple case where markups are zero $\mu = 1$. Combining with Definition 1, this term is the financial exposure of the firm to exchange rate risk $\bar{\pi}_P$

$$\frac{\text{Exports} - \text{Imports}}{\text{Total Trade}} = \gamma - \beta \propto \bar{\pi}_P.$$

In this equation the exposure term captures how financially exposed a firm is to the currency's foreign exchange rate fluctuations. When the currency of invoicing in imports and exports align, the firm is operationally hedged and able to maintain a constant desired markup. If instead $\gamma \neq \beta$, the firm is financially exposed. For example, in the UK many producers exported in the pound but imported in the euro $\beta < \gamma$. When the pound depreciated relative

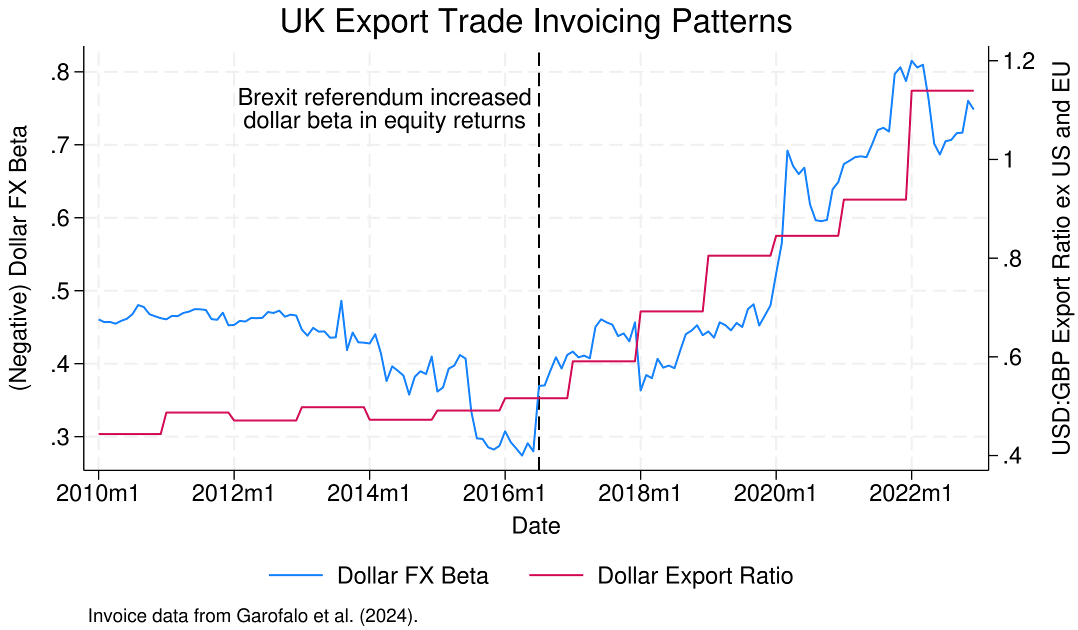

Figure 11: This graph plots the negative dollar FX beta $-\beta^{fx}$ from Equation (33). Coefficients in the regression are computed over 5-year rolling windows on monthly returns.

to the euro, import costs rose relative to export prices, reducing firm profit margins. These producers were thus exposed to price variation in profits $\bar{\pi}_P$.

The dollar's FX hedging properties also improved after Brexit because of systemic trade risk. The uncertainty surrounding existing trade treaties with the EU meant that UK exporters could lose valuable trade routes, previously made possible in the low-tariff and regulatory burden regime. A UK exporter that invoiced goods in pounds would be jointly exposed to two types of risk. First, they could be priced out of existing markets due to tariffs. Second, there was the additional possibility that their revenues (in pounds) would depreciate in value precisely when their trade routes were negatively impacted by the Brexit referendum. A British exporter could hedge the latter risk by invoicing in the dollar.

In Figure 11, I plot the hedging properties of the dollar against trade invoice shares in British exports. I compute the hedging properties by estimating the five-year rolling beta of UK equity returns (FTSE 350) to the USDGBP exchange rate and plot the negative coefficient. This is formalized by the Equation

$$r_t^{UK} = \alpha + \beta^{mkt} r_t^* + \beta^{fx} s_t + \epsilon_t \tag{33}$$

which is estimating using a five-year window on monthly pound returns. I plot the negative of the dollar FX beta $-\beta^{fx}$ to emphasize that when the dollar appreciates $s_t > 0$, the returns

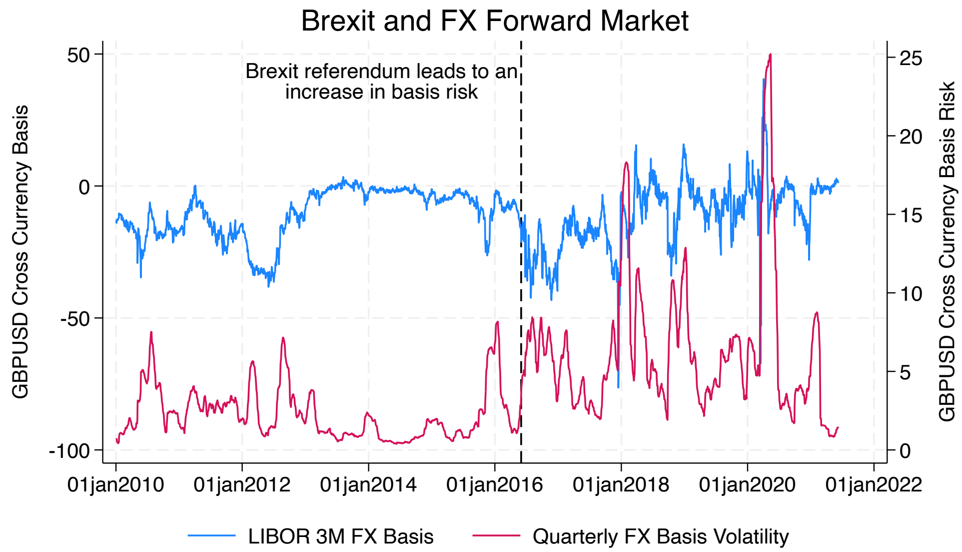

Figure 12: This figure plots the FX basis before and after Brexit and computes the quarterly volatility in the basis. A negative basis means that it is cheaper to borrow in the dollar and hedge it back into pound risk.

on UK equity tend to fall in their home currency. This beta is calculated in excess of the beta on the world market r_t^*. Consequently, it should be interpreted as a UK-specific loading on the USDGBP exchange rate. I define the world market as the MSCI world index and swap the dollar returns into the pound. I also allow for a constant α. The figure does not meaningfully vary with whether or not I control for global equity prices or allow for a constant in the linear regression.

Figure 11 shows a striking relationship between the dollar invoicing shares and the FX beta. The relationship suggests that UK firms shifted to dollar invoicing as a financial hedge against systemic trade risk. In particular, the financial hedging properties of the dollar have been trending upwards since Brexit—in tandem with the increase in dollar trade invoice shares. It is important to note that the relationship is not mechanical. Dollar invoicing in goods trade increases the dollar FX beta. That works against me in finding any negative association between the dollar invoice share and FX beta in equity returns. Hence, the true financial hedging properties of the dollar exchange rate are likely understated in this figure.

In Figure 12, I plot the FX basis over time. This figure documents that Brexit also led to dysfunction in the pound:dollar forward markets. This dysfunction caused exporters to switch from financial hedging through foreign exchange markets to synthetic hedging through the trade contract. Specifically, I show that after Brexit, the FX basis of the pound:dollar forward contract became larger and volatile. This basis represents the fair value cost of financially hedging exchange rate risk. Consequently, when the maturity of the trade contract is mismatched with the tenor of the forward contract (e.g. 1 month, 2 months, and 3 months), a volatile basis introduced uncertainty related to fluctuations in the cost of hedging.[6] Rather, by invoicing goods in the dollar, a UK exporter directly avoids this maturity mismatch and circumvents the dysfunction in the FX forward markets after Brexit. The trade contract acts as a direct mechanism for sharing pound exchange rate risk between importer and exporter, in a way that is tailored to the particularities of the trade contract.

The trade contract is an asset. Thus, the currency of invoicing decision transforms the trade contract into a synthetic foreign (or home) currency financial asset. Financial frictions jointly affect foreign exchange markets and the determination of trade invoicing, through substitution across these two types of assets. In this example, UK exporters faced greater systemic uncertainty, causing them to invoice trade in the dollar to acquire synthetic dollar assets. This synthetic dollar hedge provided valuable liquidity when trade routes deteriorated due to news of higher tariff rates with the EU. Consequently, segmentation in financial markets creates substitution away from financial hedging to trade invoicing: amplifying dollarization in the UK. However, trade invoicing has real consequences that differ from financial markets. It affects the exchange rate passthrough, which determines how exchange rate fluctuations adjust bilateral terms of trade. In contrast, financial markets enable households to save wealth across states, but do not directly impact the ability for foreign exchange rates to clear external trade imbalances.

C.3 Additional Figures for Event Study

In addition to the figures presented in the main text, Figure 13 provides two standard diagnostic plots for the synthetic control method. The first panel reports the pseudo-significance test (placebo p-values). These p-values are computed under the null that treatment assignment is random—that is, assuming the control units were untreated. Each point in the figure shows the estimated p-value as a function of the number of months since treatment. For instance, the p-value reaches zero at the 12-month horizon, implying significance at the 1 percent level.

Unlike asymptotic p-values derived from standard errors in an OLS framework, synthetic control p-values are obtained empirically through placebo reassignments. Specifically, the

[6] Additionally, when the contract is used as trade credit, there is uncertainty over when the correspondent bank approves the transaction. The bank has to validate that the trade contract was fulfilled, as it often serves an important role in contract enforcement. Delays can be considerable (Amiti and Weinstein, 2011). Alfaro, Calani and Varela (2021) find that exporters tend to enter FX forward contracts a month before the trade contract is due, resulting in long periods of exchange rate risk.

same synthetic control estimation is applied to each control unit (the "donor pool") as if it were treated. Under the assumptions of random assignment and no interference, the distribution of placebo treatment effects provides the empirical null distribution. The p-value is then defined as the share of placebo effects exceeding the estimated treatment effect for Korean RMB invoicing. Because this null distribution is discrete—reflecting a finite number of control units—the resulting p-values appear as step functions rather than smooth curves.

Following standard practice in the synthetic control literature, I restrict the placebo set to control units that exhibit an adequate preintervention fit. Specifically, I apply an RMSPE (root mean squared prediction error) filter of 5, excluding any control unit whose pre-treatment RMSPE exceeds five times that of the treated unit. This ensures that the comparison group includes only units with a similar pre-treatment trajectory. Without this filtering, poorly fitted placebo units inflate the variance of the null distribution and reduce interpretability. These units need to be removed as they violate the convex hull assumption, which states that the preintervention period needs a high quality of fit otherwise the treatment effect is poorly estimated.

Figure 13 (bottom panel) illustrates the full distribution of placebo treatment effects prior to this pruning. The black line corresponds to the estimated effect for Korea's RMB invoicing shares, while the gray lines represent placebo effects for other region-currency pairs. Many placebo units lack suitable donor matches, resulting in large pre- and post-treatment dispersion—an expected artifact of including units with poor pre-fit.

Figure 14 plots the same placebo distribution after applying the RMSPE cutoff. The treated series (CHN-RMB) stands out clearly, while the distribution of pre-treatment placebo gaps centers tightly around zero, as intended. Although there remains some post-treatment variation across control units, the treated effect is both large and persistent. One notable outlier is the EU-GBP series, which declines sharply around 2020 due to Brexit-related shifts in invoicing behavior. As discussed in Appendix Section C.2, this decline is unrelated to the CNK facility and does not affect inference.

Finally, Figure 14 also provides indirect support for the no-interference assumption. Post-treatment invoicing shares in nearby regions, such as Southeast Asia, remain stable, suggesting that China's de-dollarization policies and the Belt and Road Initiative did not materially alter invoicing patterns outside the treated bilateral relationship.

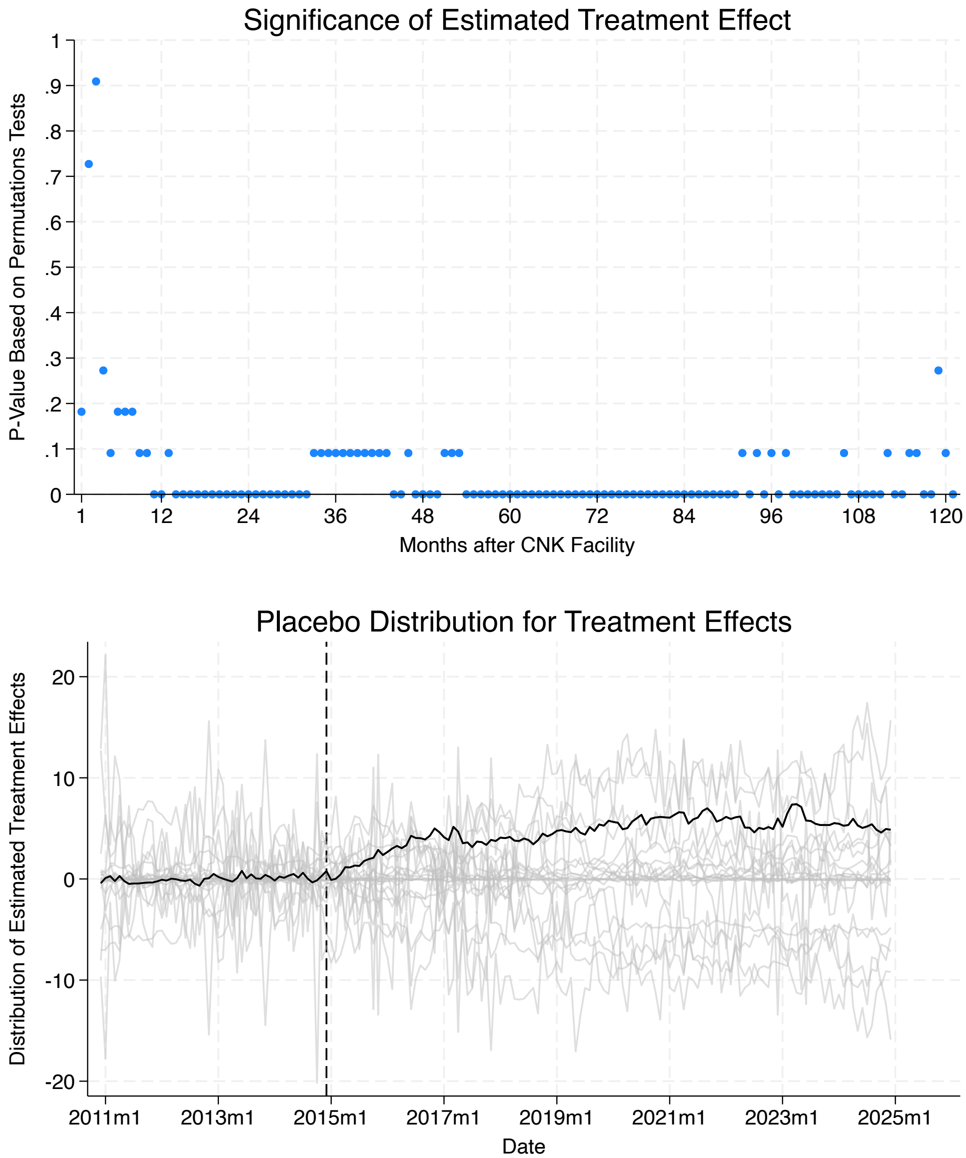

Figure 13: Diagnostic plots for the synthetic control method applied to Korean RMB invoicing shares. The top panel reports placebo-based p-values by time since treatment. The bottom panel shows treatment effects for the treated unit (black) and all placebo units (gray).

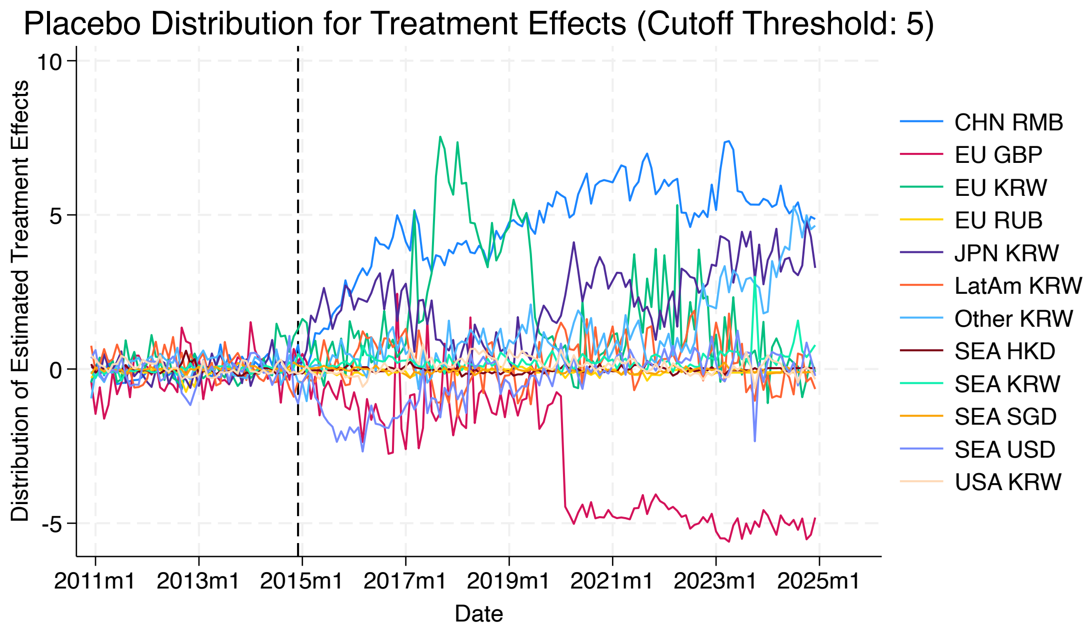

Figure 14: Distribution of treatment effects after applying the RMSPE filter. Control units are retained if their pre-treatment RMSPE is less than five times that of the treated (CHN-RMB) unit.

C.4 Examples of Master Agreements

MASTER VENDOR AGREEMENT

This Master Vendor Agreement ("Agreement") is entered into this 31 day of July, 2018, by and between TRADER JOE'S COMPANY and its subsidiaries (including Trader Joe's East Inc.). (hereinafter collectively referred to as "TJ's") and Stonegate Foods, a CALIFORNIA ("VENDOR").

SCOPE OF AGREEMENT

A. TJ's and VENDOR desire to enter into (or continue) a business relationship with each other whereby TJ's may from time to time issue purchase orders to VENDOR for certain products.

B. TJ's and VENDOR desire to set forth herein the terms and conditions that will govern their business relationship.

C. TJ's and VENDOR acknowledge that this Agreement contains the entire agreement and understanding between the Parties concerning the business relationship between TJ's and VENDOR, and any and all prior oral or written agreements or understandings between the Parties related hereto are superseded. No representations, oral or otherwise, express or implied, other than those specifically referred to in this Agreement, have been made by any party hereto, except that any prior Vendor Representation Agreement regarding VENDOR's indemnity obligations to TJ's for the use of third-party brokers or intermediaries is incorporated into this Agreement by reference, but in the event of a conflict between the two agreements, the provisions of this Agreement control.

D. TJ's and VENDOR understand that this Agreement preempts the existence of and/or overrides any previous agreements, whether express or implied, or whether oral or in writing, between TJ's and VENDOR concerning TJ's obligation and/or commitment to purchase goods from VENDOR.

⋮

TERMS AND CONDITIONS

1. Purchase Orders: It is understood and agreed that, by entering into a business relationship with VENDOR, TJ's may from time to time issue, but has not committed to issue, purchase orders to VENDOR. The parties to this Agreement understand that this Agreement and business relationship may never result in TJ's issuing any purchase orders to VENDOR and that by entering into this Agreement neither party has undertaken any obligations to the other except as expressly provided herein. VENDOR is not entering into this Agreement in reliance on the issuance of any future purchase orders by TJ's to VENDOR.

(a) This Agreement shall be incorporated by reference and made a part of any purchase order hereinafter issued by TJ's to VENDOR.

(b) Each of the terms and conditions set forth in this Agreement will apply to each purchase order unless otherwise specified by TJ's in writing.

(c) The purchase order shall set forth the description and quantity of goods, price, arrival date or schedule, payment terms, pack/size, net weight per unit, type of packaging, point of delivery, and other terms.

1

Figure 15: Example of a Master Vendor Agreement between Trader Joe's and Stonegate Foods. The contract specifies that Trader Joe's is able to make ex-post quantity demands based on the observed prices that Stonegate Foods posts at the time of sale. Source: SLS Material Contracts Corpus, SEC EDGAR database.

EXHIBIT 10.13

Execution Version

MASTER TURBINE SUPPLY AGREEMENT

BY AND BETWEEN

Guangdong Mingyang Wind Power Technology Co. Ltd.,

as "Turbine Supplier"

AND

Wind Hunter, LLC.,

as "Buyer"

Dated as of 27 November 2007

ARTICLE 2
SUPPLY OF TURBINE

2.1 Turbine Supply Commitment.

(a) Subject to Section 2.2, Turbine Supplier agrees to sell to Buyer, and Buyer agrees to purchase from Turbine Supplier, the following Turbines within the following time frames:

(i) Between the Effective Date and 31 January 2008: two (2) 1.5 MW Turbines with serial numbers 4 and 5 or higher (the "Initial Turbines").

(ii) Between 1 February 2008 and 30 July 2008: twenty (20) 1.5 MW Turbines.

2

ARTICLE 5
PRICE AND PAYMENT

5.1 Price. The Purchase Price payable for each Turbine shall be Ten Million Chinese yuan (¥10,000,000), less the Purchase Price Adjustment, and the Purchase Price for all other Supply Items and Services committed under Section 2.1 shall be as set forth on Exhibit B-3 to each Turbine Purchase Order. The cost of transportation from the port of embarkation in China to the Designated Delivery Location and the cost of insurance related such transportation shall be specified on Exhibit B-3 to each Turbine Purchase Order and shall be added to the Purchase Price for each Turbine as provided in Section 3.2. Within forty-five (45) days after the Effective Date (or such longer period as may be agreed by Turbine Supplier and Buyer), the Parties shall agree upon a price list for all Supply Items and Services (except that the price for the Turbines shall be as set forth above in this Section 5.1) and shall attach it to this Master Agreement as Exhibit B-3. The Purchase Price shall be the full price payable for each Turbine and all related Supply Items and Services.

ARTICLE 6
COMPLETION

6.1 Completion. After delivery of the Turbines and other Supply Items and Turbine Mechanical Completion has been achieved, Turbine Supplier shall achieve Completion or cause Completion to be achieved, by the Completion Dates set forth in the Delivery and Completion Schedule attached as Exhibit B-2 to the applicable TPO.

6.2 Mechanical Completion. Upon completion of the Turbine Mechanical Completion Protocol and achieving Turbine Mechanical Completion of any Turbine delivered hereunder, Buyer shall provide to Turbine Supplier a written notice of Turbine Mechanical Completion in the form attached as Exhibit E-1 ("Turbine Mechanical Completion Certificate").

6.3 Turbine Commissioning. Within forty-five (45) days after the Effective Date (or such longer period as may be agreed by Turbine Supplier and Buyer), the Parties shall agree upon a Turbine Commissioning Protocol and shall attach it to this Master Agreement as Exhibit D-2. Upon receipt of the Turbine Mechanical Completion Certificate, Turbine Supplier shall proceed with Turbine Commissioning of the Turbines pursuant to the Turbine Commissioning Protocol. Upon Completion of Turbine Commissioning, Turbine Supplier shall submit to Buyer the completed checklists for the Turbine Commissioning Protocol and the Turbine Completion Certificate set forth in Exhibit E-2.

Figure 16: Example of a Master Supply Agreement between Guangdong Mingyang Wind Power and Wind Hunter, a US based company. The contract specifies that Wind Hunter is able to determine the date at which it purchases the turbines (within a one year horizon) at a predetermined price, as defined in Section 5.1. Source: SLS Material Contracts Corpus, SEC EDGAR database.

D Macroprudential Policy: Implementing Friedman's (1953) First-Best Allocation

This section integrates the baseline model of optimal currency invoicing into a conventional small open economy model. Macro economic allocations are inefficient in this setting because export prices are sticky, creating a classic "demand externality." This demand externality drives an output gap in the tradable goods market, because the sticky price does not align with the social value of the traded good in each state.

In my model, I demonstrate that policy makers can jointly use financial taxes and conventional monetary policy to correct this demand externality. In particular, because financial taxes affect the relative cost of FX hedging, they directly influence an exporter's currency invoicing decision. I show that policy makers can leverage this insight to improve monetary policy. I propose a policy toolkit that implements producer currency invoicing with FX hedging taxes and flexible exchange rates via price targets (with commitment). I follow Friedman (1953)'s classic argument for flexible exchange rates and show that this clears the market for tradable goods and corrects the demand externality in my model.

The setting builds on Galí and Monacelli (2005). It is a small open economy model without physical capital but there exists an international financial market that facilitates risk sharing. I make two essential modifications. The first is to assume the salient pattern that international capital flows are mostly in an international currency like the dollar. The second is to assume that the tradable firms choose the optimal currency of invoicing in trade. There are two periods $t \in \{0, 1\}$. There is a continuum of symmetric small open economies $i \in [0, 1]$. Each small open economy is atomistic and has a nominal exchange rate S_{it} relative to the international currency with interest rate R_{it}. Economies are populated by a representative household, a representative nontradable firm, a tradables sector, and a local government.

D.1 Households

A representative household in each country i has preferences over tradable and nontradable consumption, as well as labor disutility. Where possible, I will suppress the notation i. The household's preferences are given by

$$\sum_{t=0}^{1} e^{-\rho t} \mathbb{E}_0 \left[U \left(C_{NT,t}, C_{T,t}, N_t \right) \right]$$

with utility U satisfying regularity assumptions. The tradable good is a CES aggregator over each countries' j variety

$$C_{T,t}^{\frac{\sigma-1}{\sigma}} = \int_0^1 C_{jT,t}^{\frac{\sigma-1}{\sigma}} dj$$

with σ being the elasticity of substitution.

Prices are nominal and denominated in the producer currency. For each period t, the household's budget constraint respects

$$\begin{aligned} & P_{T,t}C_{T,t} + P_{NT,t}C_{NT,t} + S_t B_{F,t} + B_{H,t} \\ & \leq W_t N_t + S_t \frac{R^*_{t-1}}{1+\tau^{B^F}} B_{F,t-1} + \frac{R_{t-1}}{1+\tau^{B^H}} B_{H,t-1} + \pi_t^{NT} + \int_0^1 \pi_{j,t}^T dj + T_t. \end{aligned}$$

The budget constraint states that consumption and savings decisions are financed by labor income, savings, profits from the nontradable and tradable firms, and government lump sum transfers.

I assume the market for traded goods satisfies the same price and quantity restrictions as in the baseline model, i.e. Equations (1) and (2). Specifically, prices $P_{jT,t}$ are set in advance and invoiced in a specific currency $\beta_{jT,t} \in \{0,1\}$ for traded exports from each country $j \in [0,1]$. In addition, a fraction δ of quantities are fixed in advance. This parameter is constant across all importers and exporters. Exporters are monopolists and solve the "seller's problem." Finally, the price of nontradables, produced within each country, is denoted in the producer currency by $P_{NT,t}$.

In addition, each household can save in the producer and foreign currency bond, each subject to financial taxes set by the local government. The home bond is traded exclusively by each country in zero net supply. Meanwhile, the foreign currency bond $S_t B_{F,t}$ is an internationally-traded bond that facilitates risk sharing through current accounts with an exogenously set interest rate R^*. A tax on the foreign bond is a form of capital controls, limiting foreign saving, while a tax on the home bond is expansionary, suppressing the real rate of return on the producer currency.

D.2 Firms

I simplify the analysis and assume that nontradable firms are given by a competitive representative firm. Given linear technology with labor inputs $Q_{NT,t} = A_{NT,t} L_{NT,t}$, the firm's problem is to choose a quantity

$$\pi_t^{NT} = \max_{Q_{NT,t}} \left(P_{NT,t} - \frac{W_t}{A_{NT,t}} \right) Q_{NT,t}.$$

Competition requires that the wage bill is directly tied to the price of nontradables

$$P_{NT,t} = \frac{W_t}{A_{NT,t}}. \tag{34}$$

Because nontradable firms are competitive, the wage is competitively pinned down by the price of nontradables $P_{NT,t}$ and the marginal rate of transformation $A_{NT,t}$.

Within each country, the tradable sector is made up of a continuum of symmetric tradable firms each serving a different country. Each tradable firm competes monopolistically with tradable firms from other countries, serving the same country j. Like the nontradable firm, they have linear production technology that employs labor with productivity $A_{jT,t+1}$. The government implements a labor subsidy $1 + \tau_L$ to manage the monopoly distortion. The friction in this model arises from the allocation of labor between the nontradable and tradable sector.

The tradable firm's problem fits neatly into Section 1. Given prices $P_{jT,t+1}$, quantities $Q_{jT,t+1}$, and shocks in time $t + 1$, the tradable firm profits $\pi^T_{j,t+1}$ are

$$\pi^T_{j,t+1} = \left(P_{jT,t+1} - (1 + \tau_L) \frac{W_{t+1}}{A_{jT,t+1}} \right) Q_{jT,t+1}.$$

Moreover, the profits of each tradable firm represent an infinitesimal share of the national income of the country and therefore take the SDF of the household M^i_{t+1} as given. Purchases also represent an infinitesimal part of the importing country's national income, so that the foreign household's SDF M^j_{t+1} is also taken as given. I restrict firms to either choose the producer (home) or dominant (foreign) currency, so that realized prices satisfy the nominal rigidity $P_{jT,t+1} = P_{jT,t} + \beta_{j,t} s_{t+1}$ with β being 0 or 1.

D.3 Market Clearing

In each period, the government maintains a balanced budget

$$T_t = \tau_L W_t N_t + \frac{\tau^{B^F}}{1 + \tau^{B^F}} S_t R^*_{t-1} B_{F,t-1} + \frac{\tau^{B^H}}{1 + \tau^{B^H}} R_{t-1} B_{H,t-1}.$$

While the government could borrow across time, Ricardian equivalence holds in this model. In addition, governments are responsible for setting monetary policy. Monetary policy is formulated as a target $M_t = P_{NT,t} C_{NT,t}$ with full commitment. I abstract from whether the particular implementation is done with interest rate policy or money supply, as is standard in this literature (Carvalho and Nechio, 2011).

A small open-economy equilibrium is a sequence of exogenous shocks $\{A_{NT}, A_{jT}, R^*, P_T^*\}_t$, goods markets $\{C_T, C_{NT}, Q_{jT}, Q_{NT}\}_t$, labor allocations $\{N, L_{NT}, L_{jT}\}_t$, savings decisions $\{B_H, B_F\}_t$, prices $\{S, P_{jT}, \beta_j, P_{NT,t}, R, W\}_t$, and taxes $\left\{\tau^{B^F}, \tau^{B^H}, \tau_L\right\}_t$ such that each agent optimizes, each tradable firm solves the seller's problem, the government balances its budget, and markets clear so that

$$N_t = \int_0^1 L_{jT,t}dj + L_{NT,t} \qquad \text{(Labor)}$$

$$0 = B_{H,t} \qquad \text{(Domestic Bonds)}$$

$$0 = \int_0^1 B_{jF,t}dj \qquad \text{(Foreign Bonds)}$$

$$Q_{NT,t} = C_{NT,t} \qquad \text{(Nontradables)}$$

$$P_{T,t}C_{T,t} + S_t B_{F,t} = \int_0^1 \left(P_{jT,t-1} + \beta_j s_t\right) Q_{jT,t}dj + S_t R^*_{t-1} B_{F,t-1} \qquad \text{(Trade Balance)}$$

The model is standard so implementability conditions are left in Appendix D.5.

D.4 Optimal Policy

I now set forth a policy toolkit to recover the first-best allocation. The argument is based on the insight of Friedman (1953) that in the presence of nominal rigidities, flexible exchange rates and producer currency pricing (PCP) recover efficiency. Because currency choice depends on the relative cost of FX hedging $\Delta_{ij}F$, governments can tax producer and foreign currency bonds to influence patterns in currency invoicing. In contrast, the literature often views this PCP benchmark as unattainable because invoicing is taken as exogenous. Instead, currency denomination with financial hedging suggests that regulation can manage currency invoicing.

I start by characterizing Friedman's efficient benchmark. Egorov and Mukhin (2023) demonstrates that the efficient PCP benchmark holds for Galí and Monacelli (2005)-style small open economy models.

Lemma 6 (Friedman 1953)**.** *The flexible-price equilibrium is efficient from the perspective of an individual economy and can be implemented under PCP $\beta = 0$.*

Friedman's case for flexible exchange rates boils down to the following insight: when there is a single nominal rigidity per currency, a floating exchange rate clears the market for tradable goods. Producer currency pricing is essential in this argument, otherwise the price of tradable goods does not adjust with home exchange rates.[7]

[7] The result also makes use of the symmetry assumption as the nature of the nominal rigidity among all exporting firms is identical.

This benchmark is unattainable when currency invoicing decisions are exogenous. However, with sticky quantities $\delta > 0$, segmented financial markets affect currency invoicing. Consequently, financial taxes determine if the first-best allocation is implemented.

Proposition 11. *If $\delta > 0$ and the equilibrium features stochastic exchange rates, each country i can implement the efficient PCP equilibrium iff there are*

1. *Zero foreign bond tax $\tau_i^{B^F} = 0$;*
2. *A price-level target $P_{iNT,t+1} = \frac{A_{iT,t+1}}{A_{iNT,t+1}}$;*
3. *A labor subsidy $1 + \tau_{iL} = \frac{\sigma-1}{\sigma}$; and*
4. *A home bond tax $\frac{1}{1+\tau_i^{B^H}} \leq \min_{j\in[0,1]} \left\{ \frac{1}{1+\tau_j^{B^H}} + \frac{\operatorname{Var}(s)}{2\frac{\delta}{1-\delta}\frac{\mu-1}{1+\mu\delta}} \right\}$.*

When currency invoicing reflects financial hedging $\delta > 0$, home bond taxes can implement first-best allocations. This is because currency invoicing depends on the risk preferences of households that own the exporting firms. Specifically, when policy "taxes" the domestic bond rate of return it is equivalent to increasing producer currency specialness—for example, by directly suppressing the nominal interest rate via intermediary constraints (Gabaix and Maggiori, 2015) or increasing its convenience (Jiang et al., 2024). Firms then internalize household preferences and shift their invoicing decisions towards the producer currency.

Because of the novel financial hedging mechanism, policy makers can implement producer currency pricing with a home bond tax. Goods are sold in the producer currency and, consequently, the price of tradable goods is directly controlled via home-currency price level targeting. The planner then implements the price-level target which ensures that the price of the tradable good reflects its social value in each state, i.e., the relative productivity across tradable and nontradable producers. Aligning the private and social value of the traded good ensures an efficient labor-leisure trade-off margin in each state.

The rest of this policy toolkit is standard. Foreign bonds are left untaxed to ensure efficient inter-temporal trade. A labor subsidy offsets monopoly distortions from the tradable sector. And a price-level target ensures the optimal allocation of labor across the tradable and nontradable sector. The labor-leisure margin is efficient by virtue of a competitive nontradable sector. The consumption-leisure channel is efficient because the household is not taxed across home and foreign goods.

To summarize this section: when financial markets have two instruments (i.e. producer and foreign currency bonds) but only serve to facilitate international risk sharing, a domestic bond tax can surgically implement producer currency pricing and recover first-best allocations. I want to emphasize that this is a special result developed to illustrate a broader trade-off. For example, if producer currency bonds also facilitated capital investment, the domestic bond tax would trade off PCP against underinvestment. Alternatively, the policy maker could implement PCP by holding foreign exchange reserves (Bianchi and Lorenzoni, 2022). However, an explicit trade-off comes from subsidizing foreign bond investment which distorts international risk sharing. Future work will be needed to study the general welfare trade-off of changing invoicing patterns by segmenting financial markets.

D.5 Proof of Proposition 11

For this proof, we proceed in two steps. First, we characterize the social planner's solution without nominal rigidities. It can be verified that the social planner's problem, after substituting in the home currency bond, labor, and consumption constraints is subject to a nontradables and trade balance constraint:

$$\begin{aligned} \max_{\{C_{NT,t}, C_{T,t}, L_{T,t}, L_{NT,t}, B_{F,t}\}} \sum_{t=0}^{1} \mathbb{E}_0 \left[U\left(C_{NT,t}, C_{T,t}, L_{T,t} + L_{NT,t}\right)\right] \\ \text{s.t. } P^*_{T,t} C_{T,t} + S_t B_{F,t} \leq P_{T,t} A_{T,t} L_{T,t} + S_t R^*_{t-1} B_{F,t-1} \\ A_{NT,t} L_{NT,t} \geq C_{NT,t} \end{aligned}$$

Denote the associated multipliers μ^{NT} and μ^{TB}. The first-order conditions are characterized by

$$\frac{\partial_L U_t}{\partial_{C_{NT}} U_t} = -A_{NT,t}$$

as the consumption-leisure tradeoff,

$$\frac{\partial_{C_T} U_t}{\partial_{C_{NT}} U_t} = \frac{P^*_{T,t}}{\mu^{NT}_t / \mu^{TB}_t}$$

as the expenditure switching mechanism

$$\frac{P_{T,t} A_{T,t}}{A_{NT,t}} = \frac{\mu^{NT}_t}{\mu^{TB}_t}$$

the labor allocation margin

$$\begin{aligned} \mu^{TB}_t S_t &= \mathbb{E}_t \left[\mu^{TB}_{t+1} S_{t+1} R^*_t \right] \\ \iff 1 &= \mathbb{E}_t \left[\beta \frac{\partial_{C_T} U_{t+1}}{\partial_{C_T} U_t} \frac{P^*_{T,t}}{P^*_{T,t+1}} \frac{S_{t+1}}{S_t} R^*_t \right] \end{aligned}$$

and the international risk sharing condition. These are the four conditions of efficiency.

Now, let us characterize the private solution. The household first-order conditions are

$$-\frac{\partial_L U_t}{\partial_{C_{NT}} U_t} = \frac{W_t}{P_{NT,t}}$$

as the consumption-leisure tradeoff,

$$\frac{\partial_{C_T} U_t}{\partial_{C_{NT}} U_t} = \frac{P^*_{T,t}}{P_{NT,t}}$$

as the T-NT tradeoff,

$$\begin{aligned} 1+\tau^{B^H} &= \mathbb{E}_t \left[\beta \frac{\partial_{C_T} U_{t+1}}{\partial_{C_T} U_t} \frac{P^*_{T,t}}{P^*_{T,t+1}} R_t \right] \\ 1+\tau^{B^F} &= \mathbb{E}_t \left[\beta \frac{\partial_{C_T} U_{t+1}}{\partial_{C_T} U_t} \frac{P^*_{T,t}}{P^*_{T,t+1}} \frac{S_{t+1}}{S_t} R^*_t \right] \end{aligned}$$

as the Euler equations. Given the nontradable pricing solution $P_{NT,t} = \frac{W}{A_{NT}}$, the labor-leisure and expenditure-switching margins are efficient. To match the intertemporal risk sharing condition any first-best equilibrium must satisfy $\tau^{B^F} = 0$.

What is left is the firms labor demand, pricing decision, and currency of invoice. Following the results from the section A.2, the optimal price is given by

$$P_{T,t} + \beta \mathbb{E}_t \left[s_{t+1} \right] = \frac{\sigma}{\sigma - 1} \left(1 + \tau_L\right) \mathbb{E}_t \left[\frac{W_{t+1}}{A_{T,t+1}} \right]$$

with the associated optimal passthrough

$$\beta^*/P_{T,t} = \frac{\frac{\sigma-1}{\sigma}}{\frac{\sigma-1}{\sigma}+\delta} \frac{\mathrm{Cov}_t \left(W_{t+1}/A_{T,t+1}, s_{t+1} \right)}{\mathrm{Var}_t \left(s_{t+1} \right)} + \frac{\delta \partial_x \bar{q} b_{xs} + \delta \frac{\Delta_{ij} F}{\mathrm{Var}\left(s\right)}}{\left(1-\delta\right)\left(\sigma - 1 + \delta\sigma\right)}$$

because each exchange rate is independently distributed, it follows that $b_{xs} = 0$. Thus, defining $b_\tau := \frac{\delta}{(1-\delta)(\sigma-1+\delta\sigma)\mathrm{Var}\left(s\right)}$ and $\gamma := \frac{\frac{\sigma-1}{\sigma}}{\frac{\sigma-1}{\sigma}+\delta} \frac{\mathrm{Cov}_t \left(W_{t+1}/A_{T,t+1}, s_{t+1} \right)}{\mathrm{Var}_t \left(s_{t+1} \right)}$ we have

$$\beta/P_{T,t} = \begin{cases} 0 & \frac{1+\tau_i^{B^F}}{1+\tau_i^{B^H}} < \frac{1}{b_\tau} \left(\frac{1}{2} - \gamma \right) + \frac{1+\tau_j^{B^F}}{1+\tau_j^{B^H}} \\ 1 & o.w. \end{cases}.$$

Finally, the labor allocation is given by

$$L_{T,t+1} = \frac{Q_{T,t+1}}{A_{T,t+1}}$$

where $Q_{T,t+1}$ is the demand given the realized price $P_{T,t}+\beta s_{t+1}$ for the demand curve specified in the seller's problem.

All that is left is to verify that the producer currency bond tax, monetary policy, and labor subsidies recover the optimal labor allocation under PCP, and that PCP is privately optimal for the tradables sector. The labor allocation is optimal if and only if $Q_{T,t+1}$ achieves first best. This occurs when $P_{T,t} + \beta s_{t+1}$ equals the terms of trade implied under first-best,

$$P_{T,t} + \beta s_{t+1} = \frac{W_{t+1}}{A_{T,t+1}}.$$

Combining the labor subsidy $1+\tau_L = \frac{\sigma-1}{\sigma}$ with the firm's privately optimal price, one arrives at

$$P_{T,t} + \beta \mathbb{E}_t \left[s_{t+1} \right] = \mathbb{E}_t \left[\frac{W_{t+1}}{A_{T,t+1}} \right].$$

In conjunction with monetary policy $P_{NT,t+1} = \frac{A_{T,t+1}}{A_{NT,t+1}}$ and the competitive labor condition $P_{NT,t+1} = \frac{W_{t+1}}{A_{NT,t+1}}$ this becomes

$$P_{T,t} + \beta \mathbb{E}_t \left[s_{t+1} \right] = 1 = \frac{W_{t+1}}{A_{T,t+1}}.$$

Because exchange rates s_{t+1} are stochastic, this condition holds for all states in $t+1$ iff $\beta = 0$. Thus, for $\beta = 0$ in conjunction with the first-best restriction $\tau^{B^F} = 0$, one arrives at

$$\frac{1}{1+\tau_i^{B^F}} < \frac{1}{2b_\tau} + \frac{1}{1+\tau_j^{B^H}}$$

because $\gamma = 0$ when marginal costs are constant.