# Financial Hedging and Optimal Currency of Invoicing

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**Abstract**

I develop a theory of the optimal currency choice for invoicing goods for international trade in the presence of imperfect financial hedging of currency risk. I demonstrate that the classic irrelevance result—that the cost of financial hedging does not impact the choice of currency invoicing—rests on the assumption that sellers set prices ex ante and commit to fulfill any order size ex post. I refer to this setup as sticky prices and flexible quantities. I show that when quantities are also sticky, in the sense that the order quantity is pre-specified, then financial hedging affects the optimal currency of invoicing choice. My theory of jointly sticky prices and quantities incorporates financial frictions into existing theories of real hedging. I show that this financial hedging channel is quantitatively relevant and that it generates a feedback between macroprudential policies that affect the cost of hedging, such as capital controls and the optimal currency of invoicing. I demonstrate that macroprudential policies can affect the expenditure switching properties of the exchange rate by inducing a different choice of optimal currency of invoicing.

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Trade contracts often specify a currency of invoicing. For example, a contract may denominate the price in the exporter's home currency or the importer's destination market currency. When exchange rates move, but nominal prices do not, the relative price of the traded good adjusts directly with exchange rates, thus impacting the revenue of the exporter and the importer. In aggregate, these invoicing decisions determine the expenditure-switching effect of exchange rates. Movements in the exchange rate of the invoicing currency change relative international prices and affect the demand for domestic and foreign-traded goods.

The classic theory of optimal currency invoicing posits that a monopolist can exercise additional pricing power through currency invoicing. For instance, through this channel known as *real hedging,* a seller who produces with dollar-denominated inputs will also invoice in the dollar. When the dollar appreciates, prices simultaneously appreciate with costs so that the seller maintains their desired markup on input costs. The framework has been widely adopted in the empirical analysis of currency invoicing and theories of international macroeconomics when prices are sticky.

Financial frictions do not affect the optimal currency of invoicing in the classic theory. This means that neither the availability nor the cost of financial instruments factor into the optimal invoicing decision under *real hedging*. However, financial risk is often a critical consideration in invoicing decisions of firms. For example, the US retail grocer Trader Joe's explains that "part of the intensive involvement in the buying process is to reduce risk for our vendors. We pay for that mustard in euros, their local currency now. We assume the risk of that currency fluctuation... [so that] our costs go down" (Miller & Sloan, 2021). Because the value of the trade contract is subject to FX fluctuations, the choice of invoicing currency shares financial risk. This leads to surplus that is rebated in concessionary pricing, which, in this example, allows Trader Joe's to lower costs in exchange for assuming euro/dollar risk.

In this paper, I formalize this *financial hedging* mechanism. I show that a seller will tend to invoice in the currency that insures both parties against financial risk, such as payment and default risk. I demonstrate that the mechanism is important when quantities are sticky. Based on a calibration where I use estimates of the expenditure-switching effect of exchange rates in trade, I find that a 1 ppt interest rate tax on foreign bond investments raises foreign currency invoicing shares by 50 ppt. Because trade quantities are nearly fixed in the short run, a foreign currency invoiced trade contract is a close substitute for a foreign currency bond. Consequently, sellers can invoice in foreign currency to avoid taxes on foreign currency holdings. I then incorporate my model into a small open economy setting with endogenous invoicing decisions and sticky prices. I use this setting to provide a new rationale for capital controls in the form of financial taxes, as a means of affecting invoicing patterns and increasing the efficacy of monetary policy.

This paper revisits the seminal work of Engel (2006). The classic analysis of real hedging considers a seller exporting a good and choosing ex ante whether to price the good in its home currency or that of the destination market. Ex post, the seller commits to deliver whatever quantity is demanded at the point of sale. The crux of the analysis lies in demonstrating how the choice of currency invoicing can approximate a flexible, state-contingent pricing mechanism. For instance, a seller who aims to dynamically adjust prices based on a desired markup over marginal cost can approximate this by fixing prices in the currency whose exchange rate most closely covaries with these fluctuations.

One key assumption of the classic theory of real hedging is that the quantities are flexibly determined at the time of sale. This is shared with Neo-Keynesian models à la Calvo in closed-economy macroeconomics. A voluminous literature builds on the classic theory and evaluates its predictions in the data, discusses its significance in macro models, and extends its insights to currency choices beyond those of the seller and buyer (e.g. dominant currency pricing). Throughout, the maintained result is that the external financial hedging positions of exporters and importers do not affect the invoicing currency choice in trade.

In practice, quantities are often sticky in international goods trade. Sellers often commit to the number of goods sold. In this scenario, the sellers factor in the cost of foreign exchange financial hedging in their invoicing currency choice. Practically, what this means is that for trade contracts with pre-specified quantities, financial frictions affect whether the seller intends to invoice trade in the dollar, destination market, or home currency. For example, by agreeing to purchase ten widgets at a price of 1 dollar versus 1 euro, it is as if a buyer locked in a 1:1 dollar-euro forward contract with a 10 euro notional. Consequently, when quantities are sticky because they are fixed, there is an equivalence between the trade contract and the forward exchange rate contract. The seller could then write a dollar-euro forward contract of 10 dollar notional to exactly offset the exchange rate risk induced by the trade invoicing currency choice.

Even when quantities are sticky, I find that the cost and availability of external financial hedging matters only if capital markets are also imperfect (i.e., Euler equation wedges in the foreign and domestic bond market). Precisely, if financial markets were perfect, so that sellers and buyers agreed on the price of all instruments, financial hedging would once again be irrelevant because both seller and buyer could enter equally-priced forward contracts to unwind the FX movements in prices. In contrast, consider the case of a risk-neutral seller that can hedge at no cost trading with a risk-averse buyer that is unable to access forward markets. Because financial markets are no longer perfect, a prediction of financial hedging is that the risk-neutral seller may switch from producer currency pricing (PCP) to local currency pricing (LCP) to facilitate an implicit insurance trade. Its reward for invoicing in the local currency is to capture an insurance premium in the form of a higher sale price in the local currency from the risk-averse buyer, who anticipates that the home currency exchange rate may appreciate against them.

My results nest the classic theory of real hedging. Consequently, I also assess the relative quantitative importance of real and financial hedging in an illustration in which governments impose financial taxes on foreign and domestic bond investments. These tax rates drive disagreement between the seller and buyers' price of financial instruments. The effects of these financial taxes can be substantial, such that a 1 ppt foreign bond yield difference overrides any real hedging incentive and changes the optimal currency invoicing share by 50 ppt. The effect is estimated based upon the 10 percent annualized implied volatility of at-the-money euro-dollar option derivatives and the near unitary elasticity of quantities to exchange rates in the 1 year horizon. However, context is important, since lengthening the time horizon of trade to 2 years allows quantities to adjust flexibly, more than halving the effects of financial hedging.

The setting of the numerical illustration lends itself to a discussion of the feedback between macro-prudential policy and invoicing decisions. The financial hedging channel links the optimal currency choice to wedges in the Euler equations, an object that often plays a central role in international macro policies such as financial taxes or FX intervention (Farhi & Werning, 2016; Gabaix & Maggiori, 2015; Ottonello, 2021). Indeed, financial taxes due to capital controls are one of the direct reasons why buyers and sellers might not face the same price in financial instruments. The literature focuses on FX interventions that alleviate nominal rigidities in the price of internationally traded goods, taking as given the invoicing decisions of firms. To the extent that prices are sticky, the social value of foreign-traded goods does not align perfectly with their realized price. I show that if in addition the currency invoicing is endogenous to financial hedging, there exists an optimal set of financial taxes that can implement producer currency pricing. Under limiting circumstances, this invoicing pattern implements first-best allocations as famously argued by Friedman (1953).

To deliver these insights with the simplicity and transparency of Engel (2006), I first analyze real and financial hedging using a reduced-form quantity restriction, two currencies, and Euler equation wedges. A fraction of trade is fixed in advance, and the remaining fraction determined at the time of sale. This limiting environment highlights the intuition of financial hedging models and trade contracting solutions. To generalize the results beyond fixed quantities, I develop the renegotiation-proof contracting solution with multiple currencies, and show how the optimal contract resembles the reduced-form problem in earlier sections. Because financial hedging is a form of risk sharing between a buyer and a seller, it is present in optimal trade contracts except when there is a total lack of commitment.

*Literature Review.*—The currency of invoicing for internationally traded goods has been of central importance in macroeconomic models and policy. In the presence of sticky prices, exchange rates determine relative international prices and influence consumer and producer decisions. This effect is central to Keynesian analysis (Dornbusch, 1976; Keynes, 1923; Obstfeld & Rogoff, 1995). Engel (2006) studied how an exporter optimally chooses the currency of invoicing to retain monopoly pricing power. A large literature has built on this theory of real hedging by studying producer, local, and dominant currency pricing (Amiti et al., 2022; Bacchetta & van Wincoop, 2003, 2005; Burstein & Gopinath, 2014; Corsetti & Pesenti, 2015; Gopinath & Itskhoki, 2022; Gopinath et al., 2010, 2020; Mukhin, 2022). Motivated by the presence of financial frictions in FX markets, another strand of literature assumes sticky quantities to emphasize the effects of liquidity (Krugman, 1980; Matsuyama et al., 1993), managerial frictions (Froot et al., 1993), financing costs (Coppola et al., 2023; Gopinath & Stein, 2021), and payment risk (Doepke & Schneider, 2017; Drenik et al., 2022) on currency invoicing decisions.

Evidence regarding trade contracts and the expenditure-switching role of exchange rates supports the assumption of sticky quantities. Specifically, a trade contract regularly prespecifies quantities to clarify its collateral value (Amiti & Weinstein, 2011). This enables the contract to be used in trade credit, facilitating financing and legal enforcement (Antràs & Foley, 2015). More generally, trade agreements are often imperfectly contractable, leading to "coarse" quantity variation, which may come in the form of a stepwise quantity function (Corrao et al., 2023). These decisions can be explicitly modeled in a problem with optimal supply functions (Flynn et al., 2024). By prespecifying quantities, the trade contract attenuates the classic expenditure-switching role of exchange rates on trade. Instead of flexibly adjusting demand for foreign-denominated goods, a buyer must commit to purchasing a prespecified amount as written in the contract. This reduces the aggregate response to exchange rate shocks in trade and reconciles the point estimates found in Berman et al. (2012); Devereux et al. (2017); Auer et al. (2019); Barbiero (2021); and Amiti et al. (2022). Thus, Section 2 generalizes the theory to trade contracts, demonstrating how financial hedging is a form of efficient insurance provision.

Finally, the broader implication of this paper is to reconsider some of the conclusions in the Neo-Keynesian literature on monetary policy and exchange rate management. Classic papers by Obstfeld and Rogoff (1995) and Clarida et al. (2001, 2002), as well as Galí and Monacelli (2005), assume that sellers export in their home currency. More recent literature connects this to the presence of dominant currency pricing, often in the form of invoicing in the US dollar (Basu et al., 2020; Bianchi & Lorenzoni, 2022; Bocola & Lorenzoni, 2020; Corsetti et al., 2023; Devereux et al., 2007; Egorov & Mukhin, 2023; Farhi & Werning, 2016;

Goldberg & Tille, 2009). In these papers, the currency invoicing choice is taken as given by the policymaker. I relax this assumption by allowing invoicing to depend on FX frictions. This enables me to demonstrate how financial taxes can be strategically used by regulators to incentivize home currency invoicing, which, in certain limit cases, improves allocative efficiency.

# 1 Baseline Model

This section emphasizes the basic mechanism. I focus on a two-currency case in which quantities are sticky because a fraction of the total amount sold is set in advance. The setting clarifies when and how financial hedging affects currency invoicing decisions. Generalizations and the detailed contracting setup are discussed in Section 2.

## 1.1 Setting

There are two periods. Two currencies, the home and the foreign currency, have an exchange rate denoted by $S$ in the first period and $S'$ in the second period. The home currency is the numeraire for all prices. The appreciation of the exchange rate is denoted by a small $s = S'/S - 1$, so that a $s = 2\%$ implies that the home currency depreciated two percent relative to the foreign currency. The home and foreign gross risk-free interest rates are denoted by $R$ and $R^*$. Forward contracts also exist, specifying a predetermined exchange rate $F$ and a notional amount of foreign currency to be converted into the home currency in the second period.

There is a seller $i$ and a market $j$. The market is a measure of identical buyers, each indexed by $m \in [0, 1]$. Both the seller $i$ and the market $j$ have exogenous stochastic discount factors $M^i$ and $M^j$, which are used to discount future payments. The covariance of this with the exchange rate generates a currency risk premia. Following the literature, e.g. Engel (2006), I assume they are exogenous, with the implication that the seller's currency choice does not affect the buyer and seller's marginal utility of wealth. When an agent faces a higher marginal utility of wealth in a state, it means that they discount the state less, i.e. $M$ is higher.

Formally, I write the state of the world as $x \in X$, where $x$ is a finite real random vector that occurs with probability $\mu(x)$. The expectations of random variables are taken over the state $x$, such that $\mathbb{E}[z(x)] = \int z(x)\, d\mu(x)$. All agents take $x$ as given. Consequently, the exchange rate and the SDF are both coordinates of the vector $x$, which can include a broader set of aggregate and idiosyncratic variables such as the seller's input costs or the idiosyncratic income process of each buyer.

I assume that the seller sets the terms of the trade contract. Formally, a trade contract is a unit price $P(x)$ and quantity $Q(x)$ schedule that are functions of the underlying state $x$. The seller sets the schedule to maximize the discounted profits. Given a realized state $x$, the seller's profit function $\pi$ is given by

$$\pi\left(P\left(x\right), Q\left(x\right), x\right).$$

This profit function is normalized to an outside option of 0 and satisfies the following restrictions $\partial_P \pi \geq 0$, $\partial_Q \pi > 0$, and $\pi\left(\cdot, 0, \cdot\right) = 0$. I assume it is analytic. These restrictions are satisfied by most profit functions, such as $\pi\left(P, Q, x\right) = \left(P - C\left(x\right)\right) Q$.

The seller always prefers to sell more goods at higher prices. However, the trade contract must also provide weakly positive value to each buyer. Each buyer $m$ in market $j$ derives value from trade equal to

$$V\left(Q^m\left(x\right), x\right) Q^m\left(x\right) - P\left(x\right) Q^m\left(x\right).$$

This represents the value of the good for the buyer $V$ less the total payment made, in units of home currency.[1] The value of trade is also normalized to an outside option of 0 so that the participation constraint binds when the buyer's willingness to pay equals the unit price $V\left(Q^m\left(x\right), x\right) = P\left(x\right)$. I assume that the function $V$ decreases in the quantity of trade $Q^m$ and is analytic. An example of $V$ is the standard Dixit-Stiglitz inverse demand curve $V\left(Q^m, x\right) = \mathcal{P}\left(Q^m/\mathcal{Q}\right)^{-1/\sigma}$, where the ideal price $\mathcal{P}$ and the quantity $\mathcal{Q}$ are random variables part of $x$ and $\sigma > 1$ is a constant elasticity of demand.

I now layer on two assumptions regarding the set of feasible price and quantity schedules. The first is standard from the literature (Engel, 2006).

**Assumption 1** (Price Schedule)**.** *Prices are fixed in advance but denominated in a share of the foreign currency* $\beta/P_0 \in [0,1]$

$$P\left(x\right) = P_0 + \beta s \qquad \forall x \in X. \tag{1}$$

This states that the realized price is sticky in a currency. For example, if trade is entirely invoiced in the home currency, then $\beta = 0$ so that prices do not change across states $P = P_0$. If instead trade is entirely invoiced in foreign currency, then $\beta = P_0$, so that the home

[1] I write the buyer's value in this specific form to highlight the economic mechanism of risk-sharing, while conforming the model to classic results in the literature. The results do not depend on this functional form in Section 2.

currency price changes depending on the realized exchange rate $P = P_0 \frac{S'}{S}$.[2] This restriction is standard in "firm fixed price" contracts where the price is set in advance. In absence of this restriction, a firm would set prices ex post so that $P(x)$ depends on the whole vector of $x$, and not just the coordinate representing exchange rates $s$. An example of this would be the commonly observed "cost-plus" contract, where the seller sets the unit price as some markup over the realized cost of producing the traded good.

The second assumption, specific to this paper, is the quantity schedule. I assume that a fraction of quantities are **sticky**. To highlight the financial hedging mechanism, in this section I assume that quantities are sticky because a fraction is fixed in advance. Otherwise, they are said to be **flexible**. In Section 2, I relax the assumption that quantities are fixed in advance and generalize the key theoretical results.

**Assumption 2** (Quantity Schedule)**.** *A fraction $m \in [0, \delta]$ of buyers in market $j$ fix quantities ex ante $Q^m_{ea}$ while the remaining fraction $m \in [\delta, 1]$ determines quantities ex post $Q^m_{ep}(x)$, so that the total demand is*

$$Q(x) = \delta Q^m_{ea} + (1 - \delta) Q^m_{ep}(x) \qquad \forall x \in X.$$

By setting the price and quantity, the seller internalizes the buyer's demand curve and acts as a monopolist. Hence, the seller holds the buyer to their outside option. For the fraction of buyers who determine quantities ex post, this is formalized by the state-by-state condition

$$V\left(Q^m_{ep}(x), x\right) Q^m_{ep}(x) - P(x) Q^m_{ep}(x) = 0 \qquad \forall m \in [\delta, 1], x \in X.$$

For these buyers, the seller optimally sets $Q^m_{ep}(x) = V^{-1}(P(x), x)$ as the inverse demand curve holding $x$ fixed. Using the earlier Dixit-Stiglitz demand example, ex post demand would follow the formula $Q^m_{ep}(x) = \mathcal{Q}(P(x)/\mathcal{P})^{-\sigma}$, which decreases in the realized price $P(x)$ and increases with the buyer's ideal price $\mathcal{P}$ and quantity $\mathcal{Q}$, both coordinates of the random variable $x$.

When quantities are fixed ex ante, the buyer's surplus is determined in expectation and discounted by $M^j$,

$$\mathbb{E}\left[M^j \left(V(Q^m_{ea}, x) Q^m_{ea} - P(x) Q^m_{ea}\right)\right] = 0 \qquad \forall m \in [\delta, 1].$$

The seller chooses $Q^m_{ea} \geq 0$ that satisfies this condition. This equation implicitly defines the

[2] In Engel (2006), the prices are set to be log-linear in exchange rates. Identical formulas and insights can be derived under a log-linear specification—however, in discrete-time asset pricing, it is standard to express exchange rate exposure linearly, removing the need to approximate Euler Equations.

optimal quantity $Q^m_{ea} := \bar{V}^{-1}(\mathbb{E}[M^j P])$. Unlike ex post demand $Q^m_{ep}(x) = V^{-1}(P(x), x)$, it is downward sloping in the expected discounted price $\mathbb{E}[M^j P]$. For the total quantity demanded by the destination market, the monopolist faces the demand curve

$$Q(x) = \underbrace{\delta \bar{V}^{-1}\left(\mathbb{E}\left[M^j P(x)\right]\right)}_{\text{Ex Ante Demand}} + \underbrace{(1-\delta) V^{-1}(P(x), x)}_{\text{Ex Post Demand}} \qquad \forall x \in X. \tag{2}$$

Substituting in the seller's optimal quantity schedule, the **seller's problem** is to choose a price level $P_0$ and currency denomination $\beta$, taking as given the measure over states $\mu : X \mapsto \mathbb{R}_+$, to maximize discounted profits subject to the price and quantity schedule

$$\max_{P_0, \beta} \mathbb{E}\left[M^i \pi(P(x), Q(x), x)\right], \qquad \text{s.t. Equations (1) and (2) hold.}$$

*Remark.* The baseline model assumes partially fixed quantities to emphasize classic theories of currency invoicing. While the classic literature on real hedging assumes flexible quantities $\delta = 0$, the costly financial hedging literature assumes fixed quantities $\delta = 1$. In Section 2, I show that my theoretical results do not depend on the assumption of fixed quantities or contracting with a measure of buyers. In Section 4, I show how modeling the marginal utility of wealth of each agent with exogenous SDFs captures standard FX hedging frictions, such as uninsurable buyer payment risk and seller liquidity risk.

## 1.2 Flexible Quantities

I start by analyzing currency invoicing when quantities are flexible $\delta = 0$. This benchmark mirrors the analysis of real hedging in Engel (2006). I therefore keep the exposition of this classic result streamlined.

I start by defining the concept of financial risk, expressed as the total derivatives of financial profits with respect to the realized price $P$ and the state $x$.

**Definition 1.** The **financial risk** generated by prices and shocks is given by

$$\pi_P := \partial_P \pi + \partial_Q \pi \cdot \partial_P Q \qquad \pi_x := \partial_x \pi + \partial_Q \pi \cdot \partial_x Q.$$

When prices vary, they generate financial risk through two mechanisms. First, a valuation effect $\partial_P \pi$: an increase in prices leads to a direct increase in the total value of profits, holding quantities fixed. Second, a quantity adjustment $\partial_Q \pi$ which occurs because quantities are decreasing in the realized price. I will show that this "real risk" $\partial_P Q$ scales with the share of flexible quantities. An analogous relationship holds for $\pi_x$.

*Flexible Price Solution*—In general, the seller's problem is complex because both prices and quantities are set in advance. Prices are sticky in a currency, and quantities depend on both realized and expected prices. The invoicing decision affects the realized home-currency price across states, as well as ex ante and ex post demand.

In the special case of jointly flexible prices and quantities, the posted price is state-contingent, and quantities are determined ex post. In other words, it is as if the seller, in each state of the world $x \in X$, chooses the optimal price $P(x)$ to maximize profits after observing the realized demand curve $Q(P(x), x)$. A classic first-order condition in each state $x$ characterizes the optimal flexible price $P^*(x)$[3]

$$M^i \left(\partial_P \pi + \partial_Q \pi \cdot \partial_P Q\right) \mu(x) = 0 \qquad \forall x \in X. \tag{3}$$

Consequently, when both prices and quantities are flexible, price variation does not generate financial risk $\pi_P = 0$. This means that the stochastic discount factor $M^i$ can be divided on both sides, thus dropping out. In this flexible benchmark, both the buyer and the seller's stochastic discount factors are irrelevant.

Linearizing Equation 3 around the deterministic steady state $x \to \mathbb{E}[x]$,[4] the first-order condition creates the implicit relationship

$$P^*(x) - P^*(\mathbb{E}[x]) \approx -\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} (x - \mathbb{E}[x]) \tag{4}$$

where $\bar{\pi}_{Px}$ and $\bar{\pi}_{PP}$ are second-order derivatives evaluated at the approximation point. This equation states that the optimal flexible price responds to shocks with elasticity $-\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}}$. For example, in a standard monopoly pricing problem, the firm's ideal flexible price is a desired markup $-\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}}$ over the realized marginal costs of producing the traded good, which is a coordinate of the state variable $x$.

*Sticky Price Solution*—The pricing schedule in Equation 1 restricts the price to be invoiced in a set of currencies. In other words, the home currency price can only change due to exchange rate fluctuations. This creates a "tracking error" between the sticky and flexible price, formally $\text{Var}\left(P^*(x) - P_0 - \beta s\right)$, that causes the seller to leak monopoly profits. A seller can retain its pricing power by choosing the invoicing share that best approximates the flexible price. To a second-order approximation, this is the linear projection of exchange rates onto Equation 4, a result which is often referred to by the literature as the "exchange-rate passthrough" (ERPT) onto the flexible price.

[3]It is assumed that $\pi_{PP} < 0$ whenever the probability of a state occurring $\mu(x) > 0$ in these analyses.

[4]The details of this perturbation are in Appendix A.1.

**Proposition 1** (Engel 2006)**.** *For flexible quantities $\delta = 0$, to a second-order approximation, the optimal currency denomination replicates the exchange rate passthrough (ERPT) onto flexible prices*

$$\beta^*_{\delta=0} \approx - \underbrace{\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} b_{xs}}_{\text{Real Hedging}} , \qquad \text{as } x \to \mathbb{E}[x]$$

*where* $b_{xs} := \frac{\text{Cov}(x,s)}{\text{Var}(s)}$.

This is a seminal result covered in both of the handbook chapters on currency invoicing (Burstein & Gopinath, 2014; Gopinath & Itskhoki, 2022), so I will gloss over its main implications. I provide a parameterization of it in Subsection 1.5. The key takeaway is that with flexible quantities $\delta = 0$ and sticky prices, financial markets do not affect the optimal currency invoicing decision. This is because at the approximation point, the discount factors $M^i$ and $M^j$ vanish from the seller's problem. To a second-order approximation, currency choice does not generate financial risk $\bar{\pi}_P = 0$, so incentives for risk sharing do not affect currency choice. In the next section, I show that this classic irrelevance result vanishes when FX hedging markets are imperfect and quantities are sticky.

## 1.3 Sticky Quantities

This section is the primary result of the paper: currency choice reflects financial hedging when quantities are sticky. To develop this insight, I first relate the stochastic discount factors to financial frictions using Euler equation wedges.

**Definition 2.** For each SDF $M^i$ and $M^j$, there exists an Euler Equation wedge $(\tau^i, \tau^{i*}, \tau^j, \tau^{j*})$ such that the domestic $R$ and foreign bond $R^*$ Euler Equations hold exactly,

$$1 + \tau^i := \mathbb{E}\left[M^i R\right]; \qquad 1 + \tau^j := \mathbb{E}\left[M^j R\right] \qquad \text{(Home Bond)}$$

$$1 + \tau^{i*} := \mathbb{E}\left[M^i R^* S'/S\right]; \qquad 1 + \tau^{j*} := \mathbb{E}\left[M^j R^* S'/S\right]. \qquad \text{(Foreign Bond)}$$

I define the **relative cost of FX hedging** as $\Delta_{ij}\tau = \frac{R}{R^*}\left(\frac{1+\tau^{i*}}{1+\tau^i} - \frac{1+\tau^{j*}}{1+\tau^j}\right)$.

In the model, the Euler Equation represents an agent's first-order condition from the optimal investment decision of the home and foreign currency bonds. Both SDFs are denominated in the home currency, so the payments are also written in home currency units, that is, $R$ for the home bond and $R^* \frac{S'}{S}$ for the foreign bond. Because in the background of the model there are two agents each making two investment decisions, there are four Euler Equations. When both agents are able to invest and the capital markets are perfect, the equations should hold exactly, such that $1 = \mathbb{E}[MR] = \mathbb{E}\left[MR^* \frac{S'}{S}\right]$ for both SDFs $M^i$ and $M^j$, i.e. $\tau = 0$.

I model imperfect capital markets by introducing Euler Equation wedges. An agent bond wedge $\tau$ appears on the left-hand side of each of the four first-order conditions. These wedges reconcile the agent's SDF with the observed interest rate of the bond. It concisely captures the *interest rate cost* of financial frictions. For example, $\tau^{i*} \neq 0$ captures the seller's inability to freely trade foreign currency bonds, perhaps due to transaction costs that tax the seller's purchasing price of foreign bonds. Meanwhile, $\tau^{j} \neq 0$ captures the buyer's inability to freely trade home currency bonds, perhaps due to the home country's capital controls intended to limit the inflow of foreign investment.

Finally, I define the relative cost of FX hedging $\Delta_{ij}\tau$ to capture each agent's wedge-adjusted cost of financially hedging exchange rate risk. Specifically, by short-selling the foreign currency bond and using the proceeds to invest in the home currency bond in the first period, each agent creates an investment strategy that guarantees a financial hedge in the second-period with the wedge-adjusted payoff $\frac{R}{R^*}\frac{1+\tau^*}{1+\tau} - \frac{S'}{S}$, for each unit of home currency that was initially invested.[5] This locks in a predetermined forward premium of $\frac{R}{R^*}\frac{1+\tau^*}{1+\tau}$ which may deviate from the forward premium of the market rate $F/S$. In the case in which each agent is able to frictionlessly buy and sell forwards $\frac{R}{R^*}\frac{1+\tau^*}{1+\tau} = \frac{F}{S}$ or in the case where the bond markets are perfect $\tau = 0$, the relative cost of FX hedging is zero $\Delta_{ij}\tau = 0$.

I now derive the main result of this paper. When quantities are sticky $\delta > 0$, financial frictions affect currency choice through financial hedging,

**Proposition 2.** *Let quantities be sticky $\delta > 0$. To a second-order approximation the optimal currency denomination replicates the ERPT onto flexible prices*

$$\beta_\delta^* \approx -\left( \underbrace{\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} b_{xs}}_{\textit{Real Hedging}} + \underbrace{\frac{\delta \partial_P \bar{\pi}}{\bar{\pi}_{PP}} \frac{\Delta_{ij}\tau}{\mathrm{Var}\,(s)}}_{\textit{Financial Hedging}} \right), \qquad \text{as } x \to \mathbb{E}\left[x\right].$$

*Where $\delta \partial_P \bar{\pi} = \bar{\pi}_P$ is the quantity of financial risk.*

When quantities are sticky $\delta > 0$, the optimal currency choice interacts with financial hedging through the relative cost of FX hedging $\Delta_{ij}\tau$. This proposition is a statement about cost minimization: sellers post prices in the currency that extracts the largest price concession.[6]

[5] Formally, this is derived by rearranging each of the Euler Equations in terms of the agent's certainty equivalent of the future exchange rate $\mathbb{E}\left[\frac{M}{\mathbb{E}[M]}\frac{S'}{S}\right]$, i.e. the future exchange rate under the risk-neutral measure.

[6] This endogenizes a price-to-market effect, documented in Atkeson and Burstein (2008) and Burstein et al. (2023) When a seller transacts with a buyer, the insurance premium they can charge depends on the buyer's ability to absorb the risk of exchange rates. Consequently, markups are variable by size, even under CES demand structures.

Hence, when the buyer prefers to pay in the foreign currency because it anticipates the home currency appreciating against them $\Delta_{ij}\tau > 0$, the seller switches to invoicing in the foreign currency to assume exchange rate risk. The relative cost of financing hedging $\Delta_{ij}\tau$ is in percentage points and thus scales with the amount of financial risk $\delta\partial_P\bar{\pi}$.

To provide intuition, I sketch the proof for Proposition 2 and leave the technical details in Appendix A.1. Begin by considering the solution to the flexible price and sticky quantity problem. When quantities are sticky, prices affect demand through an ex ante and ex post demand channel:

$$\frac{dQ(x)}{dP(x)} = \delta \cdot \underbrace{\frac{dQ_{ea}}{d\mathbb{E}[M^j P]} M^j \mu(x)}_{\text{Ex Ante Demand}} + (1-\delta) \cdot \underbrace{\frac{dQ_{ep}(x)}{dP(x)}}_{\text{Ex Post Demand}} \qquad \forall x \in X.$$

This ex ante demand channel is the product of two effects. The first effect is the elasticity $dQ_{ea}/d\mathbb{E}[M^j P] < 0$ which captures how an increase in the expected discounted price lowers ex ante demand. The second effect is the buyer's SDF $M^j > 0$, which quantifies how much an increase in the realized price $P(x)$ affects the discounted price. This second effect implies that the buyer lowers their ex ante demand if trade is expensive in "bad" states, i.e. when the marginal utility of wealth is high.

The first-order condition of the seller's optimal flexible price on discounted profits $\mathbb{E}[M^i\pi]$ now internalizes an effect on ex ante demand

$$M^i \cdot \underbrace{(\partial_P\pi + \partial_Q\pi \cdot \partial_P Q)}_{\text{Financial Risk } \pi_P} + M^j \cdot \underbrace{\mathbb{E}\left[M^i \partial_Q\pi \cdot \partial_{\mathbb{E}[M^j P]} Q\right]}_{\text{Ex Ante Price Elasticity}} = 0 \qquad \forall x \in X. \tag{5}$$

In the first term, the standard monopoly pricing tradeoff appears, as in Equation 3. The new term relates to ex ante demand $\partial_{\mathbb{E}[M^j P]} Q(P, \mathbb{E}[M^j P], x) < 0$ and depends on the relative SDFs of the seller $M^i$ and buyer $M^j$.[7] The key property of this equation is that it can be re-arranged to show that the seller faces financial risk from prices $\pi_P > 0$ even at the optimal flexible price. The valuation effect $\partial_P\pi$ is no longer perfectly offset by real risk $\partial_P Q$ because prices internalize an effect on ex ante demand $\partial_{\mathbb{E}[M^j P]} Q \leq 0$.

Linearizing Equation 5 around the deterministic steady state $x \to \mathbb{E}[x]$, the first-order

[7] The probability $\mu(x)$ integrates out because the flexible price in state $x \in X$ affects demand in all other states as well.

condition creates the implicit relationship

$$P^*(x) - P^*(\mathbb{E}[x]) \approx -\left( \underbrace{\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}}(x - \mathbb{E}[x])}_{\text{Markup}\times\text{Marginal Cost}} + \underbrace{\frac{\bar{\pi}_P}{\bar{\pi}_{PP}}\left(\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}\right)}_{\text{Risk-Sharing Concession/Premium}} \right). \tag{6}$$

The optimal flexible price changes with the seller and buyer's incentives to share financial risk, given by the difference in the normalized SDFs $\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}$ multiplied by the quantity of financial risk $\bar{\pi}_P$. This generalizes Equation 4 where risk-sharing dropped out due to the assumption of flexible quantities $\bar{\pi}_P = 0$. However, this risk-sharing term will also drop out in the case of perfect risk-sharing, where the seller and buyer already agree on the marginal utility of wealth in each state $M^i = M^j$.

This risk-sharing term $\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}$ is the central mechanism of this model. It implies that the seller's optimal flexible price adjusts with the buyer and seller's marginal utility of wealth, as captured by the stochastic discount factors. When risk-sharing is perfect $M^i = M^j$, the term vanishes because both parties agree on the marginal utility of wealth state-by-state. However, if there is uninsurable risk, the seller and buyer may disagree on their stochastic discount factors $M^i \neq M^j$, which creates a demand for insurance transfers. When quantities are fixed, the insurance transfer must happen through prices.

The final step of this proof is applying a linear projection of the flexible price $P^*(x)$ onto exchange rates $s$. The projection of exchange rates onto the risk-sharing term combines the Euler Equation wedges in Equation 2,

$$\text{Cov}\left(\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}, s\right) = \frac{R}{R^*}\left(\frac{1+\tau^{i*}}{1+\tau^i} - \frac{1+\tau^{j*}}{1+\tau^j}\right) := \Delta_{ij}\tau.$$

This final step rearranges the covariances in terms of the discounted expectation of exchange rates $\mathbb{E}[Ms]$. Intuitively, the equation states that the covariance of the exchange rate to the buyer and seller's relative SDFs must capture the relative cost of FX hedging. If the seller and buyer could freely trade FX hedging instruments, they would self insure against production and payment risk using the home and foreign currency bond. To the extent that they cannot perfectly insure themselves with these instruments (perhaps due to market imperfections or quantity restrictions on FX hedging contracts), the trade contract is an additional channel for risk sharing.

In the special case of perfect FX risk sharing or flexible quantities, a Modigliani-Miller irrelevance result holds. Financial hedging does not affect firm value when quantities are flexible or capital markets are perfect.

**Corollary 1** (Modigliani-Miller). *To a second order, financial hedging is irrelevant only if*

1. *Quantities are flexible* $\delta = 0$*; or*
2. *FX risk sharing is perfect* $\Delta_{ij}\tau = 0$.

## 1.4 Discrete Choice Solution

In practice, sellers often fix prices in one currency rather than choosing a fraction $\beta^*$. That is, they set prices in the home $\beta^*/P_0 = 0$ or in foreign currency $\beta^*/P_0 = 1$. The following result generalizes a threshold rule to the solution to the discrete choice problem.

**Proposition 3.** *For the home* $\beta = 0$ *versus foreign* $\beta/P_0 = 1$ *currency invoicing discrete choice problem, given* $\beta^*_\delta$ *in Proposition 2, to a second-order approximation, the threshold rule is*

$$\beta/P_0 = \begin{cases} 0 & \beta^*_\delta/P_0 < 1/2 \\ 1 & o.w. \end{cases}, \qquad as\ x \to \mathbb{E}\left[x\right].$$

*Proof.* See Appendix A.1.4 for details. □

This result was previously established by Engel (2006) for the $\delta = 0$ case. I show it holds for a more general contracting environment including the baseline model where $1 > \delta \geq 0$.[8] It states that financial frictions $\Delta_{ij}\tau$ affect the marginal switcher, for which $\beta^*_\delta/P_0$ was previously close to the one-half threshold.

The intuition for this result is that the seller's discounted profits are equal to the seller's discounted profits under the flexible pricing solution, less the "tracking error" between the sticky price and the flexible price. Deviating from the flexible price is never optimal because it satisfies a concave first-order condition in each state $x \in X$. Because $\beta^*_\delta$ was chosen to minimize this tracking error, deviating from the optimal invoicing currency share $\beta^*_\delta$ has an additional expected tracking error loss of $(\beta - \beta^*_\delta)^2 \operatorname{Var}(s) > 0$, which is symmetric on either side of $\beta^*_\delta$. Hence, the optimal discrete choice invoicing currency choice minimizes the absolute distance from the optimal currency invoicing share.

## 1.5 A Simple Example

I study a simple example to illustrate the baseline model. In this example, I specialize firm profits to have constant marginal costs to producing the traded good and assume the buyer has a constant elasticity of demand. These assumptions are standard across models of trade in small open economy settings with sticky prices. I use this model to further illustrate how

[8] Note that $\lim_{\delta\to 1} \beta^*_\delta = \pm\infty$ for $\Delta_{ij}\tau \neq 0$ and is otherwise indeterminant. Proposition 3 holds for all $\delta$ values except exactly 1. Appendix A.1.5 discusses this case—the solution is bang-bang in the sign of $\Delta_{ij}\tau$.

the cost of FX hedging is quantitatively relevant for the optimal currency of invoicing. I rely on estimates of the expenditure-switching effect of exchange rates to demonstrate that trade is mostly fixed in the short run ($\delta$ is close to 1) which quantitatively suggests that my new channel is important.

### 1.5.1 Standard Profits and Preferences

*Details of the Example*—The seller produces the traded good using a constant marginal cost production function. For the purposes of this illustration, the marginal cost curve can be described as $C = C_0(1 + \gamma \cdot s)$ where $C_0$ is the first-period home-currency price of inputs, $\gamma$ is the share of foreign inputs, and $s$ is the foreign exchange rate. When the foreign exchange rate appreciates, marginal costs increase in proportion to the share of foreign inputs. The seller's profit function is increasing in prices and decreasing in marginal costs

$$\pi(P, Q, C) = (P - C) Q.$$

The buyer's value function is given by a constant-elasticity of substitution demand curve $V(Q, \mathcal{V}) = Q^{-1/\sigma} \cdot \mathcal{V}$ where $\sigma > 1$ is the demand elasticity and $\mathcal{V} > 0$ is an exogenous willingness-to-pay index, which is a coordinate of $x$. I assume the index is uncorrelated with the exchange rate and acts as a shifter on buyer demand.[9] In this example, the state $x \in X$ is the 3-dimensional random vector consisting of the seller's marginal cost, the buyer's value index, and the percentage change in the foreign exchange rate $\langle C, \mathcal{V}, s\rangle \in \mathbb{R}^3$.

*Solution*—In Appendix A.2, I apply Proposition 2 to solve for the second-order pricing and currency invoicing decisions. Define $\mu = \frac{\sigma}{\sigma-1} > 1$ as the seller's desired markup. The optimal price

$$P_0 + \beta \mathbb{E}[s] = \mu \cdot C_0 (1 + \gamma \mathbb{E}[s]) \tag{7}$$

and currency invoicing decision is

$$\beta/P = \begin{cases} 0 & \frac{1}{1+\mu\delta} \cdot \gamma + \frac{\delta}{1-\delta} \frac{\mu-1}{1+\mu\delta} \cdot \frac{\Delta_{ij}\tau}{\mathrm{Var}(s)} \leq 1/2 \\ 1 & \text{o.w.} \end{cases}. \tag{8}$$

The firm's optimal sticky price is the expectation of the firm's optimal flexible price. In expectation, this is some markup $\mu$ over the marginal cost $C(x)$, thus deriving Equation 7. Incentives for risk sharing do not affect the expected flexible price, which in equilibrium has

[9] This is to simplify the exposition. It does not change the quantitative results. The appendix derives the full expression allowing it to be correlated.

both concessions and premiums that wash out on average.[10]

Meanwhile, the currency invoicing decision is a 1/2 threshold rule that internalizes two effects. The first effect is given by the standard force of real hedging. It depends on the foreign currency intermediate input share $\gamma \in [0,1]$ and a coefficient $\frac{1}{1+\mu\delta} \in [0,1]$. If inputs are only in the home currency $\gamma = 0$, real hedging pushes the seller towards producer currency pricing $\beta = 0$. In the classic model where $\delta = 0$, the coefficient $\frac{1}{1+\mu\delta}$ is 1, so that the canonical threshold rule is producer currency pricing $\beta/P = 0$ if and only if the foreign input cost share is less than a half $\gamma < 1/2$. This rule is driven by the seller's desire to approximate the optimal monopoly price with the currency invoicing choice $\beta$, which chooses the currency that covaries the most with marginal costs.[11]

The financial hedging mechanism manifests through the second term $\frac{\delta}{1-\delta}\frac{\mu-1}{1+\mu\delta} \cdot \frac{\Delta_{ij}\tau}{\mathrm{Var}\,(s)}$. Depending on the relative cost of FX hedging $\Delta_{ij}\tau$, the seller shifts its invoicing decision towards the currency that efficiently shares financial risk. The effect is in proportion to the term $\frac{\delta}{1-\delta}\frac{\mu-1}{1+\mu\delta}$ which vanishes with zero markups $\mu = 1$ or fully flexible quantities $\delta = 0$. Positive markups $\mu > 1$ are integral because the firm needs to earn profits to be exposed to financial risk. Formally this is captured in the earlier assumption that $\pi(\cdot, 0, \cdot) = 0$ and $\partial_Q \pi > 0$, which implies that the firm makes positive profits in equilibrium. Because the demand elasticity is finite $\sigma > 0$, the constructed example satisfies this technical restriction.

As the fraction of fixed quantities $\delta$ tends to one, the financial hedging mechanism unilaterally determines the currency of invoicing decision. In particular, the coefficient in front of $\Delta_{ij}\tau$ becomes arbitrarily positive $\lim_{\delta\to 1} \frac{\delta}{1-\delta}\frac{\mu-1}{1+\mu\delta} = +\infty$. The currency of invoicing decision responds dramatically to the cost of financial hedging when quantities are fixed because the trade contract becomes a perfect substitute with an FX forward contract. This is because the seller and buyer could enter into FX forward contracts to offset the financial risk induced by the currency invoicing decision. To the extent that the buyer and seller face different costs to entering these hedging contracts, i.e. $\Delta_{ij}\tau \neq 0$, the currency of invoicing decision should reflect the cost differential.

In theory, the determinants of currency invoicing depend on the size of markups $\mu$, the fraction of fixed quantities $\delta$, the volatility of exchange rate fluctuations $\mathrm{Var}\,(s)$, the foreign currency input share $\gamma$, and the relative cost of FX hedging $\Delta_{ij}\tau$. To bound the effect of the novel financial hedging mechanism, I now turn to evidence on the expenditure-switching

[10] Notably, it does not depend on the degree of real rigidities $\delta$. Nonetheless, there is an implicit price concession because the buyer's demanded quantity is actually higher with risk-sharing, as captured by the ex ante demand curve $Q^m_{ea} := \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right)$. This ex ante demand curve internalizes a lower price when the price $P$ covaries negatively with the stochastic discount factor $M^j$.

[11] When marginal costs are not constant, the seller internalizes how marginal costs increase when there is excess demand. Therefore, the seller will invoice in a currency that covaries positively with this excess demand.

effect of exchange rates. I show how the estimate of the elasticity of traded goods to exchange rate fluctuations informs the fraction of fixed quantities $\delta$. In particular, I find that over longer horizons of trade, the fraction of fixed quantities $\delta$ decreases (quantities are flexible in the long run). This suggests that the quantitative effect of the novel financial hedging mechanism decreases with the time duration of trade.

### 1.5.2 Bounding the Quantitative Effect of Financial Hedging

The currency of invoicing decision can respond arbitrarily to financial hedging as the fraction of fixed quantities $\delta$ approaches 1. In this section, I derive its quantitative effect using established estimates of the expenditure-switching effect of exchange rates. I find that the bounds of the term $\frac{\delta}{1-\delta}\frac{\mu-1}{1+\mu\delta}$ can vary dramatically depending on the duration of the trade contract. Although financial hedging is incredibly important in short horizons, it becomes less important in longer horizons. For this exercise, I follow the trade literature assuming that the buyer's long-run demand elasticity is $\sigma = 5$, which implies an equilibrium markup of $\mu = 1.25$ (Head & Mayer, 2014).

Formally, the expenditure-switching effect of exchange rates is the elasticity of trade with respect to exchange-rate induced price changes $dq/ds$, in the home currency of the destination market. In my model, the expenditure-switching effect is pinned down by the product of three effects

$$\underbrace{\frac{dq}{ds}}_{\text{E-S of FX}} = \underbrace{\frac{\partial q}{\partial q^{ep}}}_{1-\delta} \times \underbrace{\frac{\partial q^{ep}}{\partial p}}_{-\sigma} \times \underbrace{\left(\frac{\partial p}{\partial s} - 1\right)}_{\beta-1}. \tag{9}$$

Formally, Equation 9 states that the log change in quantities due to an appreciation of the foreign exchange rate depends on the fraction of quantities that can be adjusted $1-\delta \in [0,1]$, the demand elasticity of these quantities $\sigma > 1$, and the degree to which the unit prices change $\beta \in \{0,1\}$.

By including real rigidities in trade, my model can reconcile the expenditure switching effect of exchange rates $\frac{dq}{ds}$ with the long-run trade elasticity $\sigma$. The reason is more apparent when I rearrange Equation 9 in terms of the fraction of fixed quantities $\delta$,

$$-\frac{dq}{(1-\beta)\,ds}\Big/\sigma = 1-\delta.$$

This equation states that the share of fixed quantities $\delta$ is implied by the one minus the ratio of the long-run trade elasticity to the expenditure-switching effect in the data. When trade is flexible $\delta = 0$, the point estimates of the expenditure-switching effect and demand

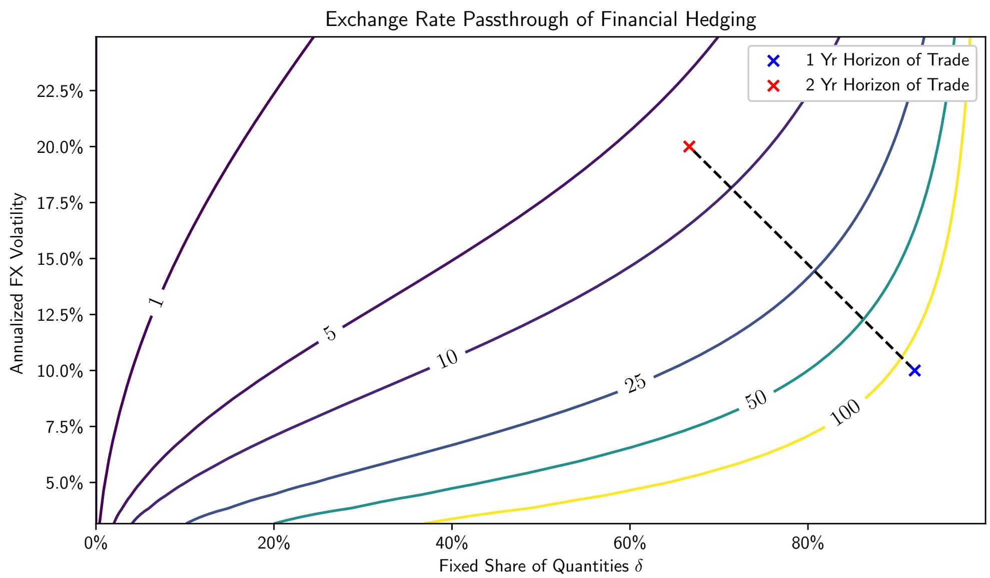


Figure 1: This contour plot shows the exchange rate passthrough of financial hedging across various specifications of exchange rate volatility and the share of sticky quantities.

elasticity are equal to each other. In contrast, when quantities do not immediately respond to exchange rate fluctuations, I interpret it as evidence of sticky quantities. My model interprets a muted expenditure-switching effect as evidence that some share of trade is fixed ex ante, for example, because trade takes time.

Amiti et al. (2022) use transaction level data to estimate these annualized elasticities. The estimates are much smaller than the long run trade elasticity $\sigma = 5$ and fall between 0.446 and 1.709, depending on the time-horizon of trade. Similar estimates are obtained in other contexts (Auer et al., 2019, 2021, 2023; Barbiero, 2021; Berman et al., 2012; Devereux et al., 2017). Consequently, using the formula above, these estimates suggest that the fixed share $\delta$ falls between 0.9 and 2/3 for the one-year and two-year time horizon, respectively. Quantities are more flexible in the long run. Exchange rate volatility also doubles for the one-year to two-year time period from 10% to 20%, since exchange rates are near random walks.[12]

Figure 1 is a contour plot where the horizontal axis is the share of fixed quantities $\delta$ and the vertical axis is the exchange rate volatility. Each contour line represents the numerical estimate of the coefficient of financial hedging $\frac{\delta}{1-\delta}\frac{\mu-1}{1+\mu\delta}$. Thus, a line equal to 50 means that a one ppt increase in the relative cost of FX hedging $\Delta_{ij}\tau$ raises the seller's ideal foreign currency share by 50 ppt, holding the effect of real hedging fixed. I plot two crosses on this plot to illustrate the bounds of my estimates. The blue cross is an upper bound on the effects

[12]Based on the at-the-money implied volatility of USDEUR European call options.

of financial hedging. There I use the real rigidity and exchange rate volatility implied by a trade contract that takes one year to settle. The other red cross is a lower bound on the effects of financial hedging, which now reflects the implied parameters of a trade contract that takes two years to settle.

The coefficient of financial hedging varies dramatically across the line that connects the estimate at the one-year and two-year horizons. As the trade contract becomes fully rigid in the one-year horizon, financial hedging becomes the driving force in currency invoicing. Foreign currency trade can be effectively used to hedge foreign currency financial risk because quantities are close to fixed. Over longer horizons, the equivalence between the trade contract and a foreign currency asset disappears. Quantities adjust when exchange rates move. Consequently, the coefficient of financial hedging shrinks from 100 to 10 across the two bounds. The key takeaway of this simple example is that financial hedging likely determines the optimal currency of invoicing for short-term trade contracts, but remains unimportant over trade contracts with longer time duration.

## 2 Contracting Microfoundation

This section microfounds the baseline model in Section 1 and studies the optimal currency of invoicing for renegotiation-proof trade contracts. A renegotiation-proof trade contract is a modeling device for capturing limited commitment, which is an empirically salient friction in international goods trade (Antràs & Foley, 2015). The particular setting satisfies a couple of purposes. The first purpose is to microfound flexible quantities with limited commitment. I show that the relative cost of FX hedging is irrelevant in currency invoicing when either party can threaten to renegotiate the contract terms, because risk sharing becomes impossible.

The second purpose is to illustrate the broader economic mechanism of financial hedging. Relative to the baseline model, I show that financial hedging does not depend on the assumption of fixed quantities. Whenever parties can commit, the seller will tilt the currency of invoicing to maximize risk sharing, so that the appropriate party assumes FX risk. The baseline model is a special instance of this where the optimal contract happens to be a fixed quantity contract. I use the contracting setup to derive a generalized version of Proposition 2 that handles broader quantity schedules. I also extend the model to include multiple currencies, as trade is often invoiced in a third currency like the US dollar. All proofs are detailed in the Appendix.

## 2.1 Setting

Fix the measure space $(\Omega, \mathcal{F}^x, \mu)$ where $\mathcal{F}^x$ is the sigma algebra generated by the $m$-dimensional state variable $x : \Omega \mapsto X \subseteq \mathbb{R}^m$. Denote $C(\Omega)$ as the set of real, bounded, and continuous functions with domain $\Omega$ and let $x \in C(\Omega)$. Trade is specified by a contract between the seller $i$ and a single buyer $j$. A contract $(P, Q) \in C(X)^2$ is a price-quantity tuple, where $X := x(\Omega)$ is the co-domain and taken as given. As before, the price function $P$ must also satisfy a nominal rigidity assumption, which is to say that $P$ is fixed in advance but indexed to one exchange rate.

$\mathcal{C}$ is the set of currencies which has the cardinality $|\mathcal{C}| = n + 1$ and thus there are $n$-bilateral pairs with the home currency. The exchange rate is modified to be a n-dimensional vector $S' : \Omega \mapsto \mathbb{R}^n_{++}$, each a coordinate of $x$. Consequently, the nominal rigidity restriction is now

$$P(\beta, x) = P_0 + \beta \cdot s \qquad \beta \in \{0, P_0 e_1, ..., P_0 e_n\}$$

where $\cdot$ is the dot product, $e_l$ is the $l$-unit vector in $\mathbb{R}^n$, and $s$ is the realized foreign currency appreciation for the $n$ bilateral pairs. For example, if $\beta = P_0 e_\$$, the trade contract is priced in the dollar.[13]

The payoffs generated by the trade contract are the profit function for the seller $\pi(P, Q, x)$ and the value of trade for the buyer $v(P, Q, x)$. I assume these functions are analytic in their arguments. This time, I focus on the case where there is exactly one buyer rather than a continuum. One can then aggregate each contract to a market of buyers, so this generalizes the baseline model. The key difference of this contracting framework is that it must be renegotiation proof. Renegotiation proof-ness is the condition that the contract is optimally never renegotiated at the time of sale, otherwise inefficient trade is implemented.

**Definition 3.** Let $\mathbb{D}^{rng} : X \rightrightarrows \mathbb{R}^2$ be the correspondence of implementable deviations. The contract $(P, Q)$ is said to be **renegotiation-proof** if $\forall x \in X$

$$v(P(x), Q(x), x) \geq \max_{\mathbb{D}^{rng}_j(x;P,Q)} v\left(\hat{P}, \hat{Q}, x\right)$$

where $\mathbb{D}^{rng}_j : X \rightrightarrows \mathbb{D}^{rng}$ is the set of deviations that weakly improve seller profits

$$\mathbb{D}^{rng}_j(x; P, Q) := \left\{\left(\hat{P}, \hat{Q}\right) \in \mathbb{D}^{rng}(x) : \pi\left(\hat{P}, \hat{Q}, x\right) \geq \pi(P(x), Q(x), x)\right\}$$

For the rest of this analysis, I take the implementable deviations correspondence $\mathbb{D}^{rng}$

[13] Here I make the outright restriction that $\beta$ is a corner solution. As with the previous section, I will first derive the unrestricted $\beta \in \mathbb{R}^n$ and then relate it to the restricted discrete choice solution $\beta \in \{0, P_0 e_1, ..., P_0 e_n\}$.

as continuous, exogenous, and inclusive of $(P(x), Q(x)) \in \mathbb{D}^{rng}(x)$. The correspondence would be exogenous if it specified a legal enforcement problem that causes the buyer to default strategically. In such a case, the deviations that the buyer could implement involve exiting the contract when the terms of trade imply a negative value relative to the outside option, which is normalized to 0.

The seller maximizes the risk-adjusted value of the contract.[14]

**Definition 4.** Given the measure space $(\Omega, \mathcal{F}^x, \mu)$ and implementable buyer deviations $\mathbb{D}_j^{rng} : X \rightrightarrows \mathbb{D}^{rng}$, the **optimal renegotiation-proof contract** $(P, Q) \in C(X)^2$ is a price-quantity pair that satisfies:

1. Profit maximization
$$\max_{(P,Q)\in C(X)^2} \mathbb{E}\left[M^i \pi(P, Q, x)\right]$$
2. Individual rationality
$$\mathbb{E}\left[M^j v(P, Q, x)\right] \geq 0$$
3. Incentive compatibility $\forall x \in X$
$$v(P(x), Q(x), x) \geq \max_{\mathbb{D}_j^{rng}(x;P,Q)} v\left(\hat{P}, \hat{Q}, x\right)$$
4. Nominal rigidity $\forall x \in X$
$$P(\beta, x) = P_0 + \beta \cdot s \qquad \text{s.t. } \beta \in \{0, P_0 e_1, ..., P_0 e_n\}.$$

## 2.2 Microfounding Quantities through Commitment

In this section, I show how the renegotiation proof contract is a microfoundation for the analysis in Section 1.2. Flexible quantities occur when the buyer lacks commitment. I now provide a formal definition for sticky quantities in the context of contracts.

**Definition 5.** Quantities are said to be **flexible** if $\forall x \in X$, there exists an implicit relationship
$$Q^* = Q(P^*(x), x).$$
Otherwise, they are **sticky**.[15]

This definition of flexible quantities subsumes the one provided in Section 1.2. In a trade contract, quantities are flexible when the seller faces a downward sloping demand curve

[14] Note that the participation constraint for the seller never binds since they exercise all bargaining power and profits satisfy limited liability.

[15] This definition of equality is in the almost surely sense.

that only depends on the realized price $P(x)$ and state variable $x$. In the baseline model, quantities were sticky when a fraction of trade was fixed, because demand also depended on the expected price $\mathbb{E}[M^j P]$. I use this definition to derive the analogue to the baseline model parameter $\delta$, which represents the fraction of buyers who fix quantities in advance.

**Proposition 4.** *If an optimal contract exists, quantities are flexible iff the buyer lacks commitment (renegotiation occurs a.s.)*

$$\mu\left(X^{cmt} := \left\{x \in X : v\left(P(x), Q(x), x\right) > \max_{\mathbb{D}_j^{rng}(x)} v\left(\hat{P}, \hat{Q}, x\right)\right\}\right) = 0. \tag{10}$$

Contract renegotiation is the microfoundation for sticky quantities. Recall that in Subsection 1.2, quantities were flexible if and only if $\delta = 0$. In this model, quantities are flexible if and only if the buyer lacks commitment $\mu\left(X^{cmt}\right) = 0$, that is to say, renegotiation occurs almost surely.[16]

Practically, buyers may lack commitment due to legal enforcement frictions. When the buyer is unable to commit, the only feasible contract is one where the seller delivers whatever is demanded at the spot price. Concretely, examine the incentive constraint. Because prices are fixed by the nominal rigidity, it follows that one can reduce the set of deviations to just quantity variation $\mathbb{D}_{jQ}^{rng}(x) := \left\{\left(\hat{P}, \hat{Q}\right) \in \mathbb{D}_j^{rng}(x) : \hat{P} = P(x)\right\}$. This means that in the states where the buyer lacks commitment, the contract optimally implements

$$v(P, Q, x) = \max_{\hat{Q} \in \mathbb{D}_{jQ}^{rng}} v\left(P, \hat{Q}, x\right) \qquad \forall x \notin X^{cmt}.$$

And by value matching, the implicit function theorem allows $Q^*$ to be rewritten as $Q(P(x), x)$ satisfying Definition 5.

However, if the contracting environment features commitment, the seller will optimally share risk. The individual rationality constraint becomes binding implying that

$$\mathbb{E}\left[M^j v(P, Q, x)\right] = 0.$$

This causes $Q^*$ to be an implicit *functional* of $P$ when it is not constrained by renegotiation. The quantities $Q^* = Q[P, x]$ then depend on the full set of price realizations.

Figure 2 visualizes a price-theory analysis of an optimal contract with renegotiated and committed states. There are two axes, the realized price (y axis) and the quantity (x axis).

[16] Evidence from Auer et al. (2023) suggests that the expenditure switching effect increases for lower income households. One explanation for this is that lower income buyers must renegotiate quantities because they are more sensitive to illiquidity.

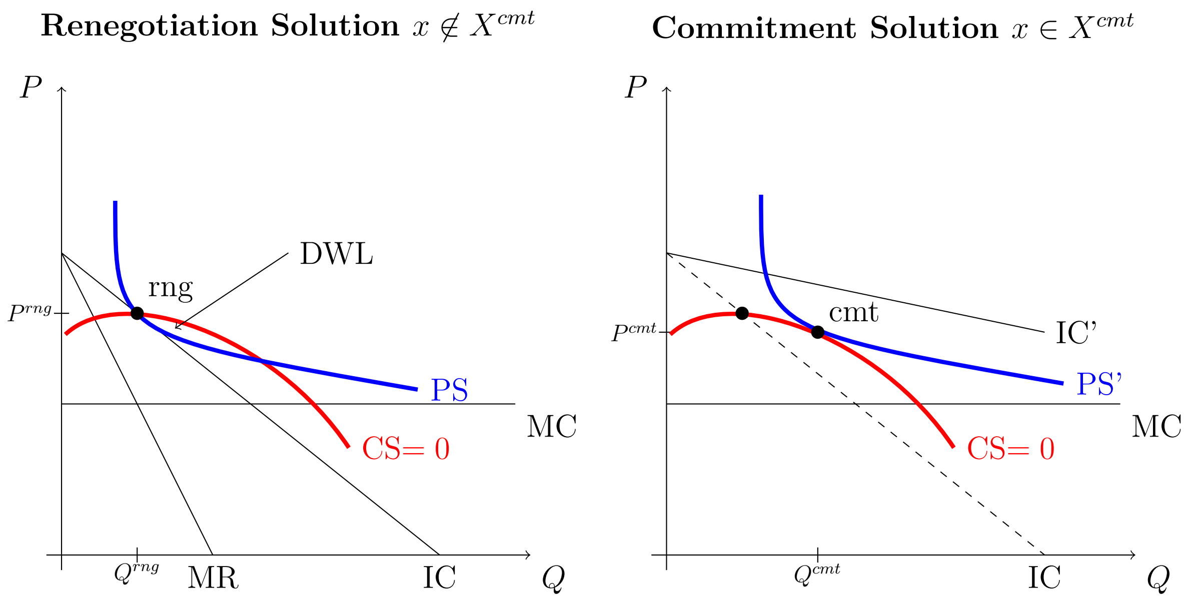


Figure 2: These graphs represent the solution to the contract across states. $PS$ and $CS$ represent the seller and buyer's ex ante indifference curves. $MR$ and $MC$ are the marginal revenue and cost curves. $IC$ and $IC'$ represents the quantity curve implied by the renegotiation constraint, which is relaxed in the right graph to allow for commitment.

The primitive is the $IC$ curve, which is a downward sloping renegotiation constraint. Renegotiation is binding on the left graph. The seller takes the renegotiation constraint as given, traces out its implied marginal revenue curve $MR$, and sets it equal to the marginal cost curve $MC$. The resulting allocation $(P^{rng}, Q^{rng})$ maximizes seller profits and satisfies the participation constraint $CS = 0$, but creates a classic monopoly deadweight loss in the oval allocations. The seller is unable to move to these points due to the threat of renegotiation. Instead, it must choose a point on $IC$.

On the right graph, the buyer and seller are able to commit to the contract terms. $IC$ shifts upwards to $IC'$, locating above the implemented price-quantity pair. Consequently, the seller finds the tangency point between the individual rationality constraint $CS = 0$ and their producer surplus curve $PS'$. This point is efficient as the seller is now able to knock-back some of the buyer's reduced surplus through generous trade terms in other states. This form of risk sharing improves allocative efficiency because it acts as a state-contingent tariff—rebating the buyer in certain states while extracting surplus when money is more valuable to the seller $M^i > M^j$.

Armed with the contracting definition of sticky quantities, I generalize the results of Proposition 2. The proposition determines the optimal currency denomination of a renegotiation proof trade contract when prices are set in a basket of currencies $\beta \in \mathbb{R}^n$.

**Proposition 5.** *Let $x \to \mathbb{E}[x]$ and $\beta \in \mathbb{R}^n$. Define the relative cost of FX hedging conditional on commitment as*

$$\Delta_{ij|X^{cmt}}\tau := \mathbb{E}\left[\left(\frac{M^i}{\mathbb{E}[M^i]} - \frac{M^j}{\mathbb{E}[M^j]}\right) s \mid X^{cmt}\right] \in \mathbb{R}^n.$$

*To a second-order approximation, the optimal currency denomination satisfies*

$$\beta^* \approx -\left(\underbrace{\frac{\bar{\pi}_{Px} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{Px}}{\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}} b_{xs}}_{\textit{Real Hedging}} + \underbrace{\frac{\bar{\pi}_P \cdot \mu\left(X^{cmt}\right)}{\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}} \Sigma^{-1}\Delta_{ij|X^{cmt}}\tau}_{\textit{Financial Hedging}}\right)$$

*where $\Sigma \in \mathbb{R}^{n\times n}$ is the variance-covariance matrix of exchange rates, $\bar{\pi}_P = \bar{v}_P \frac{\partial_Q \bar{\pi}}{\partial_Q \bar{v}}$ is the quantity of financial risk, and $\bar{v}_P = \partial_P \bar{v} + \partial_Q \bar{v} \partial_P \bar{Q}$ is the financial risk to the buyer.*

This is a generalized version of Proposition 2. The biggest change is that both real and financial hedging depend on the properties of the value of the buyer $v$. In the baseline model, the buyer value vanishes because the sticky quantities are fully fixed or flexible, so that $\bar{v}_{PP} = \bar{v}_{Px} = 0$. The seller does not need quantities to be fixed to efficiently share financial risk through currency invoicing. A more general formula will account for how quantities and the buyer value $v$ change with prices.

In addition, this formula shows a microfoundation for risk sharing. Quantities are sticky because the buyer can commit to the contract. This causes the seller to expose themselves to financial risk $\bar{\pi}_P > 0$ so that the currency denomination can extract risk sharing gains. The size of these gains are proportional to the measure of committed states $\mu\left(X^{cmt}\right)$ times the relative cost of hedging conditional on commitment $\Sigma^{-1}\Delta_{ij|X^{cmt}}\tau$. In the special case where the committed states $X^{cmt}$ are independent of financial conditions, and each state features commitment, the risk sharing gains condition down to $\Sigma^{-1}\Delta_{ij}\tau$ as before. The optimal currency of invoicing more generally depends on the share of states with commitment and vanishes when quantities are flexible.

## 2.3 Multiple Currency Choice

The nominal rigidity assumption $\beta \in \{0, P_0 e_1, ..., P_0 e_n\}$ is not satisfied by Proposition 5 because $\beta$ was not restricted to a particular currency. However, there exists an explicit relationship between the discrete choice problem and the optimal passthrough denoted by $\beta^*$. This section is the multiple-currency contracting analogue of Subsection 1.4.

**Proposition 6.** *Let $\beta \in \{0, P_0 e_1, ..., P_0 e_n\}$ satisfy the nominal rigidity. If $\beta^* < \infty$ as* *in Proposition 5, then to a second-order approximation, the solution to the discrete choice currency problem equivalently solves*

$$\beta \in \arg \min_{\bar{\beta} \in \{0, P_0 e_1, \ldots, P_0 e_n\}} \left(\bar{\beta} - \beta^*\right)^\intercal \Sigma \left(\bar{\beta} - \beta^*\right) \qquad \text{as } x \to \mathbb{E}\left[x\right],$$

*where* $\Sigma \in \mathbb{R}^{n \times n}$ *is the variance-covariance matrix of exchange rates.*

In the special case where the currency choice is binary $n = 1$, this reduces to the 1/2 threshold rule in Proposition 3. In general, there does not exist an explicit threshold rule for currency choice. This is because the seller factors in the entire covariance matrix of exchange rates $\Sigma$. If the covariance matrix is a diagonal matrix of variances, then the seller minimizes the weighted euclidean distance between the unit vector of currency $c$ $e_c$ and the optimal pass-through vector $\beta^*$, where the weight for each currency is $\text{Var}\left(s_c\right)$. When exchange rates are correlated, the seller factors in the covariances across exchange rates.[17]

# 3 Implications for Macroprudential Policy

This section integrates the baseline model of optimal currency invoicing into a conventional small open economy model. Macro economic allocations are inefficient in this setting because export prices are sticky, creating a classic "demand externality." This demand externality drives an output gap in the tradable goods market, because the sticky price does not align with the social value of the traded good in each state.

In my model, I demonstrate that policy makers can jointly use financial taxes and conventional monetary policy to correct this demand externality. In particular, because financial taxes affect the relative cost of FX hedging, they directly influence an exporter's currency invoicing decision. I show that policy makers can leverage this insight to improve monetary policy. I propose a policy toolkit that implements home currency invoicing with FX hedging taxes and flexible exchange rates via price targets (with commitment). I follow Friedman (1953)'s classic argument for flexible exchange rates and show that this clears the market for tradable goods and corrects the demand externality in my model.

The setting builds on the flexible exchange rate application in Farhi and Werning (2016). It is a small open economy model without physical capital but there exists an international financial market that facilitates risk sharing. I make two essential modifications. The first is

[17] For example, from the perspective of an exporter in New Zealand and importer in Australia, the US dollar may be the optimal currency of invoicing choice because it jointly minimizes the relative tracking error to both currencies. In particular, suppose the optimal exchange rate passthrough is the convex combination of the Australian dollar and New Zealand dollar $\beta^* = 0.5 e_{AUD} + 0.5 e_{NZD}$, then the US dollar may minimize the total tracking error $USD = \arg\min_{c \in \{AUD, NZD, USD\}} \text{Var}\left(0.5 s_{AUD} + 0.5 s_{NZD} - s_c\right)$.

to assume the salient pattern that international capital flows are mostly in an international currency like the dollar. The second is to assume that the tradable firms choose the optimal currency of invoicing in trade. There are two periods $t \in \{0, 1\}$. There is a continuum of symmetric small open economies $i \in [0, 1]$. Each small open economy is atomistic and has a nominal exchange rate $S_{it}$ relative to the international currency with interest rate $R_{it}$. Economies are populated by a representative household, a representative nontradable firm, a tradables sector, and a local government.

## 3.1 Households

A representative household in each country $i$ has preferences over tradable and nontradable consumption, as well as labor disutility. Where possible, I will suppress the notation $i$. The household's preferences are given by

$$\sum_{t=0}^{1} e^{-\rho t} \mathbb{E}_0 \left[ U \left( C_{NT,t}, C_{T,t}, N_t \right) \right]$$

with utility $U$ satisfying regularity assumptions. The tradable good is a CES aggregator over each countries' $j$ variety

$$C_{T,t}^{\frac{\sigma-1}{\sigma}} = \int_0^1 C_{jT,t}^{\frac{\sigma-1}{\sigma}} dj$$

with $\sigma$ being the elasticity of substitution.

Prices are nominal and denominated in the home currency. For each period $t$, the household's budget constraint respects

$$\begin{aligned} & P_{T,t} C_{T,t} + P_{NT,t} C_{NT,t} + S_t B_{F,t} + B_{H,t} \\ & \quad \leq W_t N_t + S_t \frac{R_{t-1}^{*}}{1 + \tau^{B^F}} B_{F,t-1} + \frac{R_{t-1}}{1 + \tau^{B^H}} B_{H,t-1} + \pi_t^{NT} + \int_0^1 \pi_{j,t}^{T} dj + T_t. \end{aligned}$$

The budget constraint states that consumption and savings decisions are financed by labor income, savings, profits from the nontradable and tradable firms, and government lump sum transfers.

I assume the market for traded goods satisfies the same price and quantity restrictions as in the baseline model, i.e. Equations 1 and 2. Specifically, prices $P_{jT,t}$ are set in advance and invoiced in a specific currency $\beta_{jT,t} \in \{0, 1\}$ for traded exports from each country $j \in [0, 1]$. In addition, a fraction $\delta$ of quantities are fixed in advance. This parameter is constant across all importers and exporters. Exporters are monopolists and solve the "seller's problem." Finally, the price of nontradables, produced within each country, is denoted in the home currency by $P_{NT,t}$.

In addition, each household can save in the home and foreign currency bond, each subject to financial taxes set by the local government. The home bond is traded exclusively by each country in zero net supply. Meanwhile, the foreign currency bond $S_t B_{F,t}$ is an internationally-traded bond that facilitates risk sharing through current accounts with an exogenously set interest rate $R^*$. A tax on the foreign bond is a form of capital controls, limiting foreign saving, while a tax on the home bond is expansionary, suppressing the real rate of return on the home currency.

## 3.2 Firms

I simplify the analysis and assume that nontradable firms are given by a competitive representative firm. Given linear technology with labor inputs $Q_{NT,t} = A_{NT,t} L_{NT,t}$, the firm's problem is to choose a quantity

$$\pi_t^{NT} = \max_{Q_{NT,t}} \left( P_{NT,t} - \frac{W_t}{A_{NT,t}} \right) Q_{NT,t}.$$

Competition requires that the wage bill is directly tied to the price of nontradables

$$P_{NT,t} = \frac{W_t}{A_{NT,t}}. \tag{11}$$

Because nontradable firms are competitive, the wage is competitively pinned down by the price of nontradables $P_{NT,t}$ and the marginal rate of transformation $A_{NT,t}$.

Within each country, the tradable sector is made up of a continuum of symmetric tradable firms each serving a different country. Each tradable firm competes monopolistically with tradable firms from other countries, serving the same country $j$. Like the nontradable firm, they have linear production technology that employs labor with productivity $A_{jT,t+1}$. The government implements a labor subsidy $1 + \tau_L$ to manage the monopoly distortion. The friction in this model arises from the allocation of labor between the nontradable and tradable sector.

The tradable firm's problem fits neatly into Section 1. Given prices $P_{jT,t+1}$, quantities $Q_{jT,t+1}$, and shocks in time $t+1$, the tradable firm profits $\pi_{j,t+1}^T$ are

$$\pi_{j,t+1}^T = \left( P_{jT,t+1} - (1 + \tau_L) \frac{W_{t+1}}{A_{jT,t+1}} \right) Q_{jT,t+1}.$$

Moreover, the profits of each tradable firm represent an infinitesimal share of the national income of the country and therefore take the SDF of the household $M_{t+1}^i$ as given. Purchases also represent an infinitesimal part of the importing country's national income, so that the foreign household's SDF $M^j_{t+1}$ is also taken as given. I restrict firms to either choose the producer (home) or dominant (foreign) currency, so that realized prices satisfy the nominal rigidity $P_{jT,t+1} = P_{jT,t} + \beta_{j,t} s_{t+1}$ with $\beta$ being 0 or 1.

## 3.3 Market Clearing

In each period, the government maintains a balanced budget

$$T_t = \tau_L W_t N_t + \frac{\tau^{B^F}}{1+\tau^{B^F}} S_t R^*_{t-1} B_{F,t-1} + \frac{\tau^{B^H}}{1+\tau^{B^H}} R_{t-1} B_{H,t-1}.$$

While the government could borrow across time, Ricardian equivalence holds in this model. In addition, governments are responsible for setting monetary policy. Monetary policy is formulated as a target $M_t = P_{NT,t} C_{NT,t}$ with full commitment. I abstract from whether the particular implementation is done with interest rate policy or money supply, as is standard in this literature (Carvalho & Nechio, 2011).

A small open-economy equilibrium is a sequence of exogenous shocks $\{A_{NT}, A_{jT}, R^*, P^*_T\}_t$, goods markets $\{C_T, C_{NT}, Q_{jT}, Q_{NT}\}_t$, labor allocations $\{N, L_{NT}, L_{jT}\}_t$, savings decisions $\{B_H, B_F\}_t$, prices $\{S, P_{jT}, \beta_j, P_{NT}, R, W\}_t$, and taxes $\left\{\tau^{B^F}, \tau^{B^H}, \tau_L\right\}_t$ such that each agent optimizes, each tradable firm solves the seller's problem, the government balances its budget, and markets clear so that

$$N_t = \int_0^1 L_{jT,t} dj + L_{NT,t} \qquad \text{(Labor)}$$

$$0 = B_{H,t} \qquad \text{(Domestic Bonds)}$$

$$0 = \int_0^1 B_{jF,t} dj \qquad \text{(Foreign Bonds)}$$

$$Q_{NT,t} = C_{NT,t} \qquad \text{(Nontradables)}$$

$$P_{T,t} C_{T,t} + S_t B_{F,t} = \int_0^1 \left(P_{jT,t-1} + \beta_j s_t\right) Q_{jT,t} dj + S_t R^*_{t-1} B_{F,t-1} \qquad \text{(Trade Balance)}$$

The model is standard so implementability conditions are left in Appendix A.5.

## 3.4 Optimal Policy

I now set forth a policy toolkit to recover the first-best allocation. The argument is based on the insight of Friedman (1953) that in the presence of nominal rigidities, flexible exchange rates and producer currency pricing (PCP) recover efficiency. Because currency choice depends on the relative cost of FX hedging $\Delta_{ij}\tau$, governments can tax home and foreign currency bonds to influence patterns in currency invoicing. In contrast, the literature often views this PCP benchmark as unattainable because invoicing is taken as exogenous. Instead, currency denomination with financial hedging suggests that regulation can manage currency invoicing.

I start by characterizing Friedman's efficient benchmark. Egorov and Mukhin (2023) demonstrates that the efficient PCP benchmark holds for Galí and Monacelli (2005)-style small open economy models.

**Lemma 1** (Friedman 1953)**.** *The flexible-price equilibrium is efficient from the perspective of an individual economy and can be implemented under PCP $\beta = 0$.*

Friedman's case for flexible exchange rates boils down to the following insight: when there is a single nominal rigidity per currency, a floating exchange rate clears the market for tradable goods. Producer currency pricing is essential in this argument, otherwise the price of tradable goods does not adjust with home exchange rates.[18]

This benchmark is unattainable when currency invoicing decisions are exogenous. However, with sticky quantities $\delta > 0$, segmented financial markets affect currency invoicing. Consequently, financial taxes determine if the first-best allocation is implemented.

**Proposition 7.** *If $\delta > 0$ and the equilibrium features stochastic exchange rates, each country $i$ can implement the efficient PCP equilibrium iff there are*

1. *Zero foreign bond tax $\tau_i^{B^F} = 0$;*
2. *A price-level target $P_{iNT,t+1} = \frac{A_{iT,t+1}}{A_{iNT,t+1}}$;*
3. *A labor subsidy $1 + \tau_{iL} = \frac{\sigma-1}{\sigma}$; and*
4. *A home bond tax $\frac{1}{1+\tau_i^{B^H}} \leq \min_{j\in[0,1]} \left\{ \frac{1}{1+\tau_j^{B^H}} + \frac{\mathrm{Var}(s)}{2\frac{\delta}{1-\delta}\frac{\mu-1}{1+\mu\delta}} \right\}$.*

When currency invoicing reflects financial hedging $\delta > 0$, home bond taxes can implement first-best allocations. This is because currency invoicing depends on the risk preferences of households that own the exporting firms. Specifically, when policy "taxes" the domestic bond rate of return it is equivalent to increasing home currency specialness—for example, by directly suppressing the nominal interest rate via intermediary constraints (Gabaix & Maggiori, 2015) or increasing its convenience (Jiang et al., 2024). Firms then internalize household preferences and shift their invoicing decisions towards the home currency.

Because of the novel financial hedging mechanism, policy makers can implement producer currency pricing with a home bond tax. Goods are sold in the home currency and, consequently, the price of tradable goods is directly controlled via home-currency price level

[18]The result also makes use of the symmetry assumption as the nature of the nominal rigidity among all exporting firms is identical.

targeting. The planner then implements the price-level target which ensures that the price of the tradable good reflects its social value in each state, i.e., the relative productivity across tradable and nontradable producers. Aligning the private and social value of the traded good ensures an efficient labor-leisure trade-off margin in each state.

The rest of this policy toolkit is standard. Foreign bonds are left untaxed to ensure efficient inter-temporal trade. A labor subsidy offsets monopoly distortions from the tradable sector. And a price-level target ensures the optimal allocation of labor across the tradable and nontradable sector. The labor-leisure margin is efficient by virtue of a competitive nontradable sector. The consumption-leisure channel is efficient because the household is not taxed across home and foreign goods.

To summarize this section: when financial markets have two instruments (i.e. home and foreign currency bonds) but only serve to facilitate international risk sharing, a domestic bond tax can surgically implement producer currency pricing and recover first-best allocations. I want to emphasize that this is a special result developed to illustrate a broader trade-off. For example, if home currency bonds also facilitated capital investment, the domestic bond tax would trade off PCP against underinvestment. Alternatively, the policy maker could implement PCP by holding foreign exchange reserves (Bianchi & Lorenzoni, 2022). However, an explicit trade-off comes from subsidizing foreign bond investment which distorts international risk sharing. Future work will be needed to study the general welfare trade-off of changing invoicing patterns by segmenting financial markets.

# 4 Examples of Financial Hedging

In this section, I show how my model, which assumes exogenous stochastic discount factors for buyers and sellers, captures standard mechanisms in the financial hedging literature. I demonstrate this by showing how the relative cost of FX hedging can be rewritten in terms of financial frictions such as uninsurable payment risk, trade financing costs, transaction taxes, and seller liquidity risk. I use these examples to revisit empirical patterns in currency invoicing that I interpret as evidence of financial hedging.

## 4.1 Buyer Payment Risk

Buyer payment risk is a common source of financial hedging in currency denomination (Doepke & Schneider, 2017; Drenik et al., 2022). A theory of money characterizes currencies as a unit of account—so that, when the buyer is liable for payment, all else equal she prefers to pay in a currency that covaries with her wealth.

Consider a risk-neutral seller and a potentially risk-averse buyer. The risk-neutral seller has a constant discount factor $M^i = R^{-1}$. On the other hand, the risk averse buyer has an indirect utility function $U$ which takes as arguments their wealth and the state of the world. The buyer invests in a home and foreign currency deposit, earning interest rates $R$ and $R^*$. Their share in the foreign currency deposit is given by $\theta$, so that the maximization is written

$$\max_{\theta} \mathbb{E}\left[U\left(W, x\right)\right] \qquad \text{s.t. } W \leq W_0\left[R + \theta\left(R^* S'/S - R\right)\right].$$

This objective models buyer payment risk. As the realized wealth $W$ falls, the buyer's indirect utility can become arbitrarily negative due to an Inada condition. Denote the coefficient of relative risk aversion as $RRA := -\frac{\partial_{ww} U \cdot W}{\partial_w U}$.

**Proposition 8.** *To a second-order approximation, the buyer's optimal LC savings share is*

$$\theta \approx \frac{\mathbb{E}\left[R^* S'/S - R\right]}{RRA \cdot \text{Var}\ (s)} \qquad \text{as } x \to \mathbb{E}\left[x\right].$$

*So a sufficient statistic for the relative cost of FX hedging is given by*

$$\Delta_{ij}\tau := R^{-1} \cdot \underbrace{RRA \cdot \text{Var}\ (s)}_{\text{Risk Aversion}} \cdot \underbrace{\theta}_{\text{LC Share}} \qquad \text{as } x \to \mathbb{E}\left[x\right].$$

*Proof.* See Appendix A.6. □

In this application of the theory, currency invoicing reflects the buyer's local currency wealth shares. Intuitively, the trade contract diversifies the buyer's investment risk and leads to an efficient price concession. This predicts a positive empirical relationship between a currency's import share and the buyer's cross-currency deposit shares.

The mapping between the theory and data is measurable through aggregate patterns. To the extent that trade contracts share financial risk, aggregate pricing patterns should reflect aggregate foreign currency debt investments. Aggregate import currency invoicing patterns are available for 70 countries in 2019 due to a dataset constructed by the World Bank (Boz et al., 2022). Meanwhile, buyer cross-currency investment shares can be measured using the BIS Location Banking Statistics, which report the quantity of bank liabilities in various currencies. Specifically, I focus on the liabilities where the counterparty is a nonbanking institution, for example capturing retail firms and households with preexisting deposits. There are substantial limitations to using BIS data since the foreign banking sector does not capture other debt securities that a buyer may purchase, such as corporate or sovereign debt. The representativeness of the BIS data therefore biases the relationship towards zero, and works against finding a pattern.

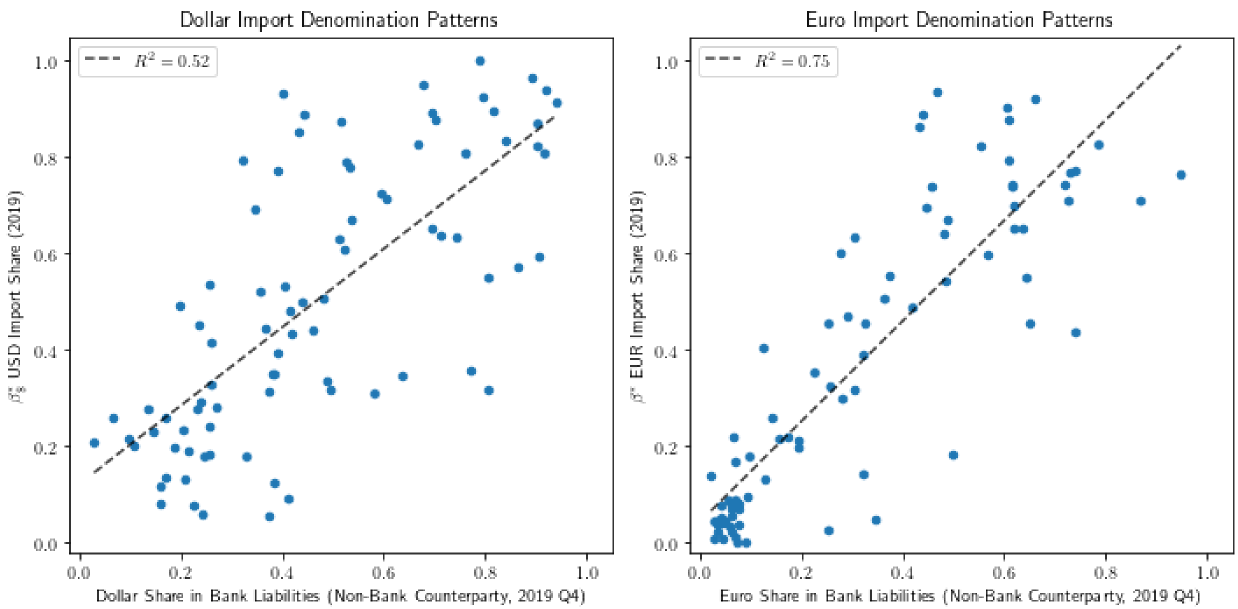


Figure 3: This scatter plot compares the cross-country currency denomination patterns versus the buyer's domestic banking sector foreign currency liabilities shares. The x-axis represents one measurement of financial hedging.

Figure 3 plots a positive relationship between currency denomination shares and the buyer's foreign currency banking deposit shares. This pattern suggests that buyer payment risk is relevant in the currency denomination of international goods trade. A larger Dollar and Euro share in the buyer's bank deposit sector suggests that more depositors hold their investments in foreign currency. Intuitively, the trade contract diversifies this investment risk by choosing the dollar and the euro. The relationship is strong but incomplete—theory anticipates that real hedging explains the residual variation in currency choice.

## 4.2 Trade Financing Costs

When the seller borrows across different currencies, cross-currency trade financing frictions measure financial hedging. Intuitively, currency choice in the trade contract alleviates financing frictions, since it serves as collateral against a seller's net debt position. This process is often referred to as "trade credit" and is vital in international goods trade (Amiti & Weinstein, 2011; Bocola & Bornstein, 2023; Iacovone et al., 2019; Ronci, 2004).

I now augment the model to include cross-currency financing. The exporter must borrow $B$ to invest in capital $I$. They can borrow through their local bank that charges an upward sloping interest rate curve $R : \mathbb{R} \mapsto \mathbb{R}_{++}$ and $R^* : \mathbb{R} \mapsto \mathbb{R}_{++}$ with constant elasticity $\partial_b r \geq 0$

and $\partial_{b^*} r^* \geq 0$. The profit maximization problem is now

$$\max_{P_0,\beta^\intercal,B,I} \mathbb{E}\left[M^i\left(\underbrace{\pi\left(P,Q,I,x\right)}_{\text{Gross Income}} - \underbrace{R^*\left(B^*\right)\frac{S'}{S}B^*}_{\text{Foreign CC}} - \underbrace{R\left(B\right)B}_{\text{Domestic CC}}\right)\right]$$

subject to the total level of borrowing at least exceeding the level of investment

$$B + B^* \geq I.$$

Otherwise, the stated problem is identical.

Modeling the cross-currency financing problem reveals a sufficient statistic for the value of financial hedging. Reading off the seller's first-order condition,

**Proposition 9.** *Assume the buyer's risk preferences are determined by the local bank,*

$$\mathbb{E}\left[\frac{M^j}{\mathbb{E}\left[M^j\right]}S'\right] = F.$$

*A sufficient statistic for the relative cost of FX hedging is given by*

$$\Delta_{ij}\tau := \underbrace{\frac{1+\partial_b r}{1+\partial_{b^*} r^*}}_{\textit{Market Depth}} - \underbrace{\frac{F}{S}\cdot\frac{R^*}{R}}_{\textit{CIP}}.$$

*Proof.* See Appendix A.7. □

For each currency, two characteristics govern the degree of financial hedging: deviations in covered interest rate parity and relative market depth. Covered interest rate parity is the no-arbitrage condition that domestic interest rates $R$ should equal foreign interest rates $R^*$ after hedging exchange rate risk $F/S$.[19] In the model proposed by Gopinath and Stein (2021), covered interest rate parity is a sufficient statistic for the choice of currency, because it is precisely the shadow price of a costly collateral constraint. On the other hand, the relative market depth $\frac{1+\partial_b r}{1+\partial_{b^*} r^*}$ reflects the impact of sourcing capital across domestic and foreign capital markets. In the model proposed by Coppola et al. (2023), market depth is a sufficient statistic for currency choice, because it measures the monopsony rents collected from denominating a firm's balance sheet in a liquid currency.

Financial hedging is related to these two features through revealed preference. Intuitively, the size of these statistics reflects a financial friction that leads to an Euler equation wedge.

[19] In recent years, there has been evidence covered interest rate parity deviations due to banking regulation (Du & Schreger, 2016; Du et al., 2018; Keller, 2024).

Thus, the Euler equation wedge $\Delta_{ij}\tau$ can be measured by characteristics of the foreign exchange market, such as the size of CIP and market depth differentials. The characteristics then reveal the cost of financial frictions, such as unmodeled capital controls, search costs, taxes, leverage constraints, and collateral requirements, which force the firm to leave money on the table.

## 4.3 FX Transaction Costs

Another measurement of financial hedging is total transaction costs. Transaction costs come in the form of bid-ask spreads, creating a wedge between the hypothetical market exchange rate and the realized market exchange rate. Swoboda (1969), Krugman (1980), and Rey (2001) associate the theory of a dominant currency with transaction costs, which may arise due to search frictions as in Matsuyama et al. (1993).

This section formalizes the relationship between bid-ask spreads and financial hedging. A trade contract overcomes transaction costs, since the choice of currency denomination circumvents the buyer and seller going through a costly intermediary. To formally characterize transaction costs, the market exchange rate now depends on whether a currency is being bought or sold and which currency it is being converted to and from.

**Definition 6.** The bid-ask of an exchange rate $S_c$, for currency $c \in \mathcal{C}$, and for agents $a \in \{i, j\}$ are scalars satisfying

$$S_{ac}^{ask} := S_c \left(1 + \tau_{ac}^{ask}\right) \qquad S_{ac}^{bid} := S_c \left(1 + \tau_{ac}^{bid}\right) \qquad \forall a \in \{i, j\}, c \in \mathcal{C}$$

with the **bid-ask spread** defined as $\Delta_{bid/ask}\tau_{ac} := \frac{1+\tau_{ac}^{bid}}{1+\tau_{ac}^{ask}} \geq 1$.

The bid-ask spreads are a direct measure of transaction costs for a currency. $\tau_a$ is a percentage spread between the hypothetical market exchange rate $S$ and agent $a$'s quoted exchange rate $S_a$. Differences across agents $\tau_i \neq \tau_j$ reflect the fact that the seller $i$ and buyer $j$ may ultimately convert cashflows into different currencies or through different intermediaries. Differences across the bid and ask $\tau_a^{ask} \leq 0 \leq \tau_a^{bid}$ reflect the fact that the sale of a currency (ask) faces a markdown while its purchase (bid) faces a markup.

In the trade contract, the buyer receives exchange rate risk while the seller is short the exchange rate risk. This leads to two Euler equations which reflect the buyer and seller's risk preferences over quoted exchange rates.

**Proposition 10.** *For the transfer between buyer to seller, the Euler equation satisfies*

$$1 = \underbrace{\mathbb{E}\left[M^i \frac{S_{ic}^{bid'}}{S_{ic}^{ask}} R_c^*\right]}_{\text{Seller Long FX}}; \qquad 1 = \underbrace{\mathbb{E}\left[M^i \frac{S_{ic}^{ask'}}{S_{ic}^{bid}} R_c^*\right]}_{\text{Buyer Short FX}}.$$

*Assume the bid-ask spread is constant over time. A sufficient statistic for the relative cost of FX hedging is given by*

$$\Delta_{ij}\tau := \underbrace{\frac{\Delta_{bid/ask}\tau_i}{\Delta_{bid/ask}\tau_i^*}}_{\text{Seller Net T-Costs}} - \underbrace{\frac{\Delta_{bid/ask}\tau_j^*}{\Delta_{bid/ask}\tau_j}}_{\text{Buyer Net T-Costs}}.$$

*Proof.* See Appendix A.8. □

Financial hedging privileges currencies that minimize transaction costs. Increasing the foreign bid-ask spread always reduces the gains from denominating in a foreign currency $\frac{d\Delta_{ij}\tau}{d\Delta_{bid/ask}\tau^*} < 0$. Per Goldberg and Tille (2008) and Krugman (1980), three types of pricing regimes occur:

1. Producer currency pricing, since $\Delta_{bid/ask}\tau_i = 1$
2. Local currency pricing, since $\Delta_{bid/ask}\tau_j^* = 1$
3. Dominant currency pricing, since $\Delta_{bid/ask}\tau_i^{\$} \downarrow 1$ and $\Delta_{bid/ask}\tau_j^{\$} \downarrow 1$.

PCP minimizes the seller's transaction costs; LCP minimizes the buyer's transaction costs; and DCP represents a second-best alternative for both agents.

Transaction costs are a prominent aspect of exchange rate regimes prior to the 21st century (Vries, 1987, 1996). The source of these transaction costs came in the form of parallel exchange rate markets. Although the government would maintain an "official" exchange rate, sanctioned for a particular economic activity or subject to quantity restrictions, a "parallel" exchange rate would develop through unofficial channels. These rates were at a premium depending on the relative supply and demand of the money in question.

Figure 4 is a cross-country scatterplot of the parallel FX premium relative to the USD export shares found in Boz et al. (2022). The parallel FX premium data are taken from Ilzetzki et al. (2019), which was manually collected from the World Currency Yearbook. The year 1998 was chosen to maximize data availability, since the World Currency Yearbook was discontinued afterwards, while currency invoicing in trade generally increases in coverage across the years.

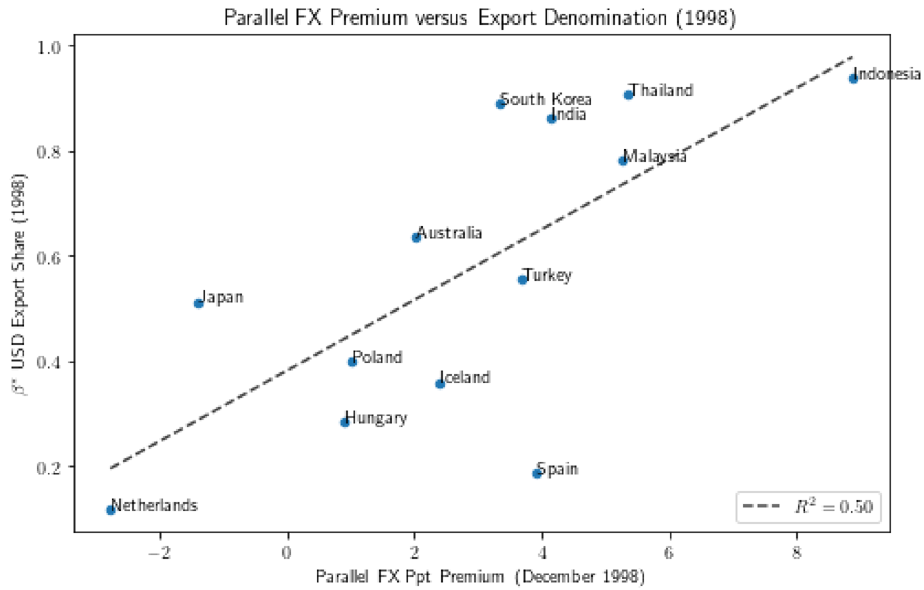


Figure 4: This scatter plot compares the cross-country currency denomination patterns versus each exporting country's parallel FX premium, measured as the percentage point spread in the Dollar exchange rate.

Consistent with the insights of this subsection, this figure shows a positive correlation between the FX premium and the USD export share. The FX premium is a transaction tax on the unofficial and official exchange rate of the market. Therefore, countries that have a larger FX premium, such as Indonesia, Thailand, and Malaysia, all have larger USD export shares. Although this is indirect evidence of the mechanism, it provides a concrete application relating currency choice to financial frictions.

## 4.4 Firm Liquidity

Firm liquidity also measures the value of financial hedging. Intuitively, the seller can use the trade contract to hedge against default. For example, if the seller is poorly capitalized, it may choose to price in the producer currency to avoid exposing itself to exchange rate movements. In contrast, if the seller is well-capitalized, it may instead price in the local currency to capture risk-sharing gains.

Following Froot et al. (1993), I microfound firm liquidity using a costly state verification setting. Costly state verification was developed by Townsend (1979) to rationalize debt as the optimal contract in the corporate finance environment. In this model, I find that the value of financial hedging is measured by the firm's probability of default and the distress price of exchange rates. This is consistent with the theoretical insights in Bruno and Shin (2015) and Eren et al. (2023).

There are now three agents in the model: the seller $i$, the buyer $j$ and the lender $\ell$. All agents are now risk neutral and discount time at rate 1.[20] Consequently, risk and time preferences are identical so that financial hedging occurs only endogenously. The seller $i$ raises capital from the lender $\ell$ at time $t$ equal to $I$. In return, the seller repays $D$ to the lender $\ell$ in the next period. The seller can also default on its promise. If the seller defaults, the lender must engage in costly state verification paying a cost $c \geq 0$ to verify and seize the seller's profits. In this setting, it is known that optimal contracts are debt and equity. Thus, $D$ is a constant.

The objective of the seller is to maximize

$$\max_{P_0, \beta^{\intercal}, I, D} \underbrace{\mathbb{E}\left[\pi\left(P, Q, I, x\right) - D\right]^{+}}_{\text{Expected Residual Profits}}.$$

The seller jointly chooses prices, currency denomination, quantities, investment, and the level of debt. When investing, the seller reduces future costs but increases the costs of default. When profits fall below the default threshold $D$, the seller must forfeit the profits to the lender and receive nothing.

The lender only participates if the net present value of the future debt obligation $D$ exceeds the investment cost. This creates a standard investment constraint,

$$I \leq \mathbb{E}\left[\underbrace{D 1_{\pi > D}}_{\text{No Default}} + \underbrace{\left(\pi - c\right) 1_{\pi \leq D}}_{\text{Default}}\right].$$

The constraint states that the total capital offered by the lender $\ell$ is less than or equal to the expected payment. All other constraints–incentive, participation, and nominal rigidity–are kept constant in this analysis.

**Proposition 11.** *Define $G$ as the CDF of the equilibrium profits and $g$ as its corresponding PDF. Let $D^*$ be the equilibrium debt level and define the "distress price" of exchange rates,*

$$\underbrace{s_{\pi \leq D}}_{\textit{Distress Price}} := \underbrace{\left(1 - G\left(D^*\right)\right) \mathbb{E}\left[s \mid \pi^* = D^*\right]}_{\textit{FX Conditional at Default Threshold}} + \underbrace{G\left(D^*\right) \mathbb{E}\left[s \mid \pi^* < D^*\right]}_{\textit{FX Conditional in Default}}.$$

*A sufficient statistic for the relative cost of FX hedging is given by*

$$\Delta_{ij}\tau := \underbrace{\frac{cg\left(D^*\right)}{1 - G\left(D^*\right) - cg\left(D^*\right)}}_{\textit{Default Intensity}} \times \underbrace{s_{\pi \leq D}}_{\textit{Distress Price}}.$$

[20]Risk neutrality and absence of time discounting are invoked to simplify expressions.

*Proof.* See Appendix A.9. □

The firm is unable to financially hedge due to costly state verification. Consequently, the cost of hedging measures the cost of exchange rate volatility, which is captured by the intensity of the default and the price of the distress. The default intensity is a scalar which captures how much the lender up-charges the cost of capital due to default. Consequently, it depends on both the cost of verification $c \geq 0$ and the intensity of the default at the equilibrium level of debt $g(D^*) \geq 0$.

The distress price reflects whether the currency appreciates during firm default. Unlike the default intensity term, $s_{\pi \leq D}$ determines whether each pair of currencies is likely to appreciate or depreciate. For example, the seller is likely to prioritize the producer currency, since these remain flat during global business cycle downturns. However, the intensity of this incentive is regulated by the cost of default $cg(D^*)$.

Empirically, this suggests that in transaction-level pricing data, a larger seller should price more in the local currency because they can easily absorb financial risk. Both Amiti et al. (2022) and Devereux et al. (2017) find evidence of a significant connection between size and currency choice, even after controlling for theories of real hedging. In the data, larger Belgium firms are more likely to export in the local currency to extra-EU countries than smaller firms, within time, destination, and controlling for competitor pricing decisions. These firms share risk efficiently and use the local currency pricing decision to extract price concessions from buyers, which may be small retailers or households.

# 5 Conclusion

This paper develops a theory of currency choice in international goods trade with imperfect foreign exchange markets. The paper begins by reexamining the canonical theory of currency choice, which separates real and financial hedging. I show that since this theory assumes flexible quantity contracts, the financial hedging incentive drops out, and foreign exchange markets become irrelevant. If instead quantities are sticky and capital markets are segmented, currency choice reflects financial hedging incentives. This formalizes how financial conditions such as buyer payment risk, financing costs, transaction costs, and firm liquidity empirically affect currency choice. It provides a novel toolkit for macroprudential policy to affect invoicing patterns and reduce inefficiencies.

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# A Proofs and Further Details on the Theory

## A.1 Details on Propositions 2 and 3

To deliver the results on optimal currency invoicing, we formalize the problem as a functional and define the perturbation. This extends a classic projection result to the solution of second-order contracting problems. It simplifies proofs and allows us to handle the general contracting problem in Section 2 with ease.

### A.1.1 Definitions

By visual inspection, the reduced-form and general contracting problem are special cases of the following convex program.

**Definition 7.** Fix $(\Omega, \mathcal{F}^x, \mu)$ as the measure space. Denote $C(\Omega)$ as the set of bounded-continuous real-valued functions with the domain $\Omega$ endowed with the metric defined by the sup norm. Let $x \in C(\Omega)$ and $a, b \in C(X)$ where $X := x(\Omega)$.

Define the functional $S : C(X)^2 \times C(\Omega) \mapsto \mathbb{R}$ with $f$ being infinitely differentiable (in the Frechet sense) so that $S$ is defined as

$$S[a, b, x] := \int_\Omega f(a, b, x)\, d\mu$$

subject to a set of infinitely differentiable (in the Frechet sense) constraints

$$0 \in G[a, b, x]$$

The convex program is a set of functions $(a^*, b^*) \in C(X)^2$ which maximizes $S[a, b, x]$ subject to satisfying the constraints.

To avoid redundancy, throughout we assume that the differential operator $\delta$ is applied in the Frechet sense. It is known that this generalizes the derivative as defined over the real vector space. Moreover, the following fact will be useful.

**Fact 1.** *The vector spaces $C(X)$ and $C(\Omega)$ with the sup norm metric are Banach spaces.*

This fact is often used in functional analysis. In addition, I require the following set of technical conditions to hold, ensuring the existence of a KKT representation of the infinite-dimensional convex program and well-defined higher-order derivatives.

**Lemma 2.** *Suppose $b^*[a, x]$ satisfies the Robinson constraint qualification*

$$0 \in int\left\{G[a, b, x] + \delta_b G[a, b, x](b - b^*) : b \in C(X) \times C(\Omega)\right\}$$

*so that the set of Lagrange multipliers is non-empty. If the constraint is of the form* $0 \in \int_\Omega g(a,b,x)\,d\mu = G[a,b,x]$ *for some infinitely differentiable function* $g$*, then it follows that there is some neighborhood* $\bar{\mathcal{A}}, \bar{\mathcal{B}} \subset C(X)$ *and* $\bar{X} \subset C(\Omega)$*such that* $b^*[a,x]$ *is of class* $C^\infty$ *and characterized by some equivalent function* $b^*(a,x,\lambda^*[a,x])$*. Moreover, the objective can be rewritten as*

$$S_b[a,x] = \int_\Omega \tilde{f}(a,x,\lambda^*[a,x])\,d\mu$$

*and is also of class* $C^\infty$ *over the domain* $C(X) \times C(\Omega) \mapsto \mathbb{R}$.

*Proof.* Given the Robinson constraint qualification holds for $G$, the convex program admits a nonempty, convex, and bounded set of Lagrange multipliers (Theorem 3.9 Bonnans and Shapiro 2000). The Lagrangian multipliers can be represented as such,

$$\mathcal{L}[a,x] := \min_{\lambda \geq 0} \max_{b \in C(X)} \int_\Omega [f(a,b,x) - \lambda g(a,b,x)]\,d\mu.$$

For any local maximum, the following first-order conditions are necessary and sufficient

$$\begin{aligned} \partial_b f(a,b^*,x) - \lambda^* g(a,b^*,x) &= 0 \\ \int_\Omega g(a,b^*,x)\,d\mu &= 0. \end{aligned}$$

The Implicit Function Theorem (for Banach spaces) as applied to these equations implies that the optimal $b$ can be rewritten as $b^*(a,x,\lambda^*)$ and $\lambda^* = \lambda^*[a,x]$, and moreover for some neighborhoods $\bar{\mathcal{A}} \subset C(X)$ and $\bar{X} \subset C(\Omega)$, the composite $b^*[a,x]$ is of class $C^\infty$. In this neighborhood, we may rewrite

$$S_b[a,x] = \int_\Omega f(a, b^*(a,x,\lambda^*[a,x]), x)\,d\mu$$

which is also of class $C^\infty$ over the domain $C(X) \times C(\Omega) \mapsto \mathbb{R}$ due to the chain rule. By substitution, it follows that there exists some function $\tilde{f}(a,x,\lambda^*[a,x]) = f(a,b^*(a,x,\lambda^*[a,x]),x)$. □

### A.1.2 Projection Method

**Lemma 3.** *Assume a local optimum exists in some open set* $a^* \in \bar{\mathcal{A}} \subset C(X)$ *and the constraint satisfies the representation in Lemma 2. If the function* $a \in \bar{\mathcal{A}}$ *is a multilinear that takes the form* $a(x) = a_0 + a_{x_l} \cdot x_l$ *where* $a_{x_l} \in \mathbb{R}^A$ *and* $x_l$ *is a subset of the coordinates* *of $x \in X$, the objective achieves*

$$S_b [a, x] - S_b [a^*, x] = \frac{1}{2} A [\mathbb{E}[x]] \cdot \mathbb{E}\left[\left\{a_0 + a_{x_l}^{\intercal} x_l - \left(a^* (\mathbb{E}[x]) + \partial_x a^* (\mathbb{E}[x])^{\intercal} (x - \mathbb{E}[x])\right)\right\}^2\right]$$
$$o\left(\|x - \mathbb{E}[x]\|^2\right), \qquad \textit{as } x \to \mathbb{E}[x].$$

*where $A[x]$ is defined in the proof. Moreover, when the second-order condition holds $A[\mathbb{E}[x]] \leq 0$, the maximizing $a_{x_l} \in \mathbb{R}^A$ is given by*

$$a_{x_l} = \text{Var}\ (x_l)^{-1} \text{Cov}\ (x_l, x)\, \partial_x a^* (\mathbb{E}[x]).$$

*Proof.* The fundamental theorem of calculus applies to $S_b$ since $C(X)$ is a Banach space:

$$\begin{aligned} S_b [a, x] &= S_b [a^*, x] + \int_0^1 \frac{dS_b [a^* + (a - a^*) t, x]}{dt} dt \\ &= S_b [a^*, x] + \int_0^1 \frac{\delta S_b}{\delta (a^* + (a - a^*) t)} [a, x]\, \delta (a(x) - a^*(x))\, dt \end{aligned}$$

where the second line follows by way of explicitly expressing the differential. Using $\delta S_b [a^*, x; h] = 0$ from the fact that $a^*$ is a local maximum, we get

$$\begin{aligned} \frac{\delta S_b}{\delta (a^* + (a - a^*) t)} (x) &= \frac{\delta S_b}{\delta a^*} (x) + \int_0^t \frac{\delta^2 S_b}{\delta (a^* + (a - a^*) \tau)^2} (x)\, \delta (a(x) - a^*(x))\, d\tau \\ &= \int_0^t \frac{\delta^2 S_b}{\delta (a^* + (a - a^*) \tau)^2} (x)\, \delta (a(x) - a^*(x))\, d\tau. \end{aligned}$$

Note that because $a^*$ is a local maximum, the variation $\delta^2 S_b$ must be bounded. Combining

$$S_b [a, x] - S_b [a^*, x] = \int_0^1 \int_0^t \frac{\delta^2 S_b}{\delta (a^* + (a - a^*) \tau)^2} (x)\, \delta^2 (a(x) - a^*(x))\, d\tau dt.$$

Conjecture that the optimal solution of $a$ satisfies $a(\mathbb{E}[x]) = a^*(\mathbb{E}[x])$.

Let $x \to \mathbb{E}[x]$. It is known that the Taylor Approximation theorem holds for infinitely differentiable functions on Banach Spaces (Theorem 5.6.1 Cartan and Maestro 2017). Define the linearized difference between the solution $a^*$ and the multilinear control $a$ as

$$\Delta a(x) = a_0 + a_{x_l} \cdot x_l - a^* (\mathbb{E}[x]) - \partial_x a^* (\mathbb{E}[x]) \cdot (x - \mathbb{E}[x])$$

then the second-order approximation is given by

$$S_b[a,x] - S_b[a^*,x] = \frac{1}{2}\left(\lim_{x\to\mathbb{E}[x]} \frac{\delta^2 S_b}{(\delta a)^2}(x)\right)\Delta a(x)^2 + o\left(\|x - \mathbb{E}[x]\|^2\right)$$

where $\|\cdot\|$ is the sup norm over the space of continuous functions.

Denote the function

$$\hat{a}(x') = \begin{cases} a^*(x(\omega)) & \{\omega\in\Omega : x(\omega)\neq x'(\omega)\} \\ a(x(\omega)) & \Omega\setminus\{\omega\in\Omega : x(\omega)\neq x'(\omega)\} \end{cases}.$$

Rewrite the total second-order derivative in terms of the pointwise second-order derivatives, which are defined by differentiating with $\hat{a}$. Integrating across states, this becomes

$$\left(\lim_{x\to\mathbb{E}[x]} \frac{\delta^2 S_b}{(\delta a)^2}[a,x]\right)\Delta a(x)^2 = \int_\Omega\int_\Omega \frac{\delta^2 S_b}{\delta\hat{a}(x')\,\delta\hat{a}(x'')}\Delta\hat{a}(x')\,\Delta\hat{a}(x'')$$

where each pointwise derivative is given by

$$\frac{\delta^2 S_b}{\delta\hat{a}(x')\,\delta\hat{a}(x'')}[a,x] = \partial_{aa}\tilde{f}d\mu 1_{x'=x''} + 2\partial_{\lambda a(x')}\tilde{f}\partial_{a(x'')}\lambda d\mu + \left(\int\partial_\lambda\tilde{f}d\mu\right)\partial_{a(x')a(x'')}\lambda + \left(\int\partial_{\lambda\lambda}fd\mu\right)\partial_{a(x')}\lambda\partial_{a(x'')}\lambda.$$

By totally differentiating the constraint $\int g(a, b^*(a,x,\lambda^*[a,x]), x)\,d\mu$ it follows that

$$\partial_{a(x')}\lambda^* = -\frac{\partial_a g(x') + \partial_b g(x')\,\partial_a b^*(x')}{\int\partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu}d\mu$$

and moreover

$$\partial_{a(x')a(x'')}\lambda^* = -\frac{\partial_{aa}g(x')\,1_{x'=x''} + \partial_{ab}g(x')\left(\partial_a b^*(x')\,1_{x'=x''} + \partial_\lambda b^*(x')\,\partial_{a(x'')}\lambda\right)}{\int \partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu} d\mu$$
$$- \frac{\partial_{ba}g(x')\,\partial_a b^*(x')\,1_{x'=x''} + \partial_{bb}g(x')\,\partial_a b^*(x')\left(\partial_a b^*(x')\,1_{x'=x''} + \partial_\lambda b^*(x')\,\partial_{a(x'')}\lambda\right)}{\int \partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu} d\mu$$
$$- \frac{\partial_b g(x')\left(\partial_{aa}b^*(x')\,1_{x'=x''} + \partial_{a\lambda}b^*(x')\,\partial_{a(x'')}\lambda\right)}{\int \partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu} d\mu$$
$$+ \frac{\partial_a g(x') + \partial_b g(x')\,\partial_a b^*(x')}{\left[\int \partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu\right]^2} d\mu \left[\partial_{ba}g(x)\,\partial_\lambda b^*(x) + \partial_b g(x)\,\partial_{\lambda a}b^*(x)\right] d\mu.$$
$$+ \frac{\partial_a g(x') + \partial_b g(x')\,\partial_a b^*(x')}{\left[\int \partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu\right]^2} d\mu \int \partial_{bb}g(x)\left[\partial_a b^*(x)\,1_{x=x''} + \partial_\lambda b^*(x)\,\partial_{a(x'')}\lambda\right] d\mu$$
$$+ \frac{\partial_a g(x') + \partial_b g(x')\,\partial_a b^*(x')}{\left[\int \partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu\right]^2} d\mu \int \partial_b g(x)\,\partial_{\lambda\lambda}b^*(x)\,\partial_{a(x'')}\lambda d\mu$$

Importantly, the limits of both the first and second-derivative can be expressed as

$$\lim_{x\to\mathbb{E}[x]} \partial_{a(x')}\lambda^* = \zeta\left[\mathbb{E}[x]\right] d\mu(x')$$
$$\lim_{x\to\mathbb{E}[x]} \partial_{a(x')a(x'')}\lambda^* = \xi\left[\mathbb{E}[x]\right] d\mu(x')\,d\mu(x'') + 1_{x'=x''}\psi\left[\mathbb{E}[x]\right] d\mu(x')$$

for scalar functionals $\zeta, \xi, \psi$. Plugging this into the equation above, we get

$$\lim_{x\to\mathbb{E}[x]} \frac{\delta^2 S_b}{\delta\hat{a}(x')\,\delta\hat{a}(x'')}[a,x] = A\left[\mathbb{E}[x]\right] 1_{x'=x''}d\mu(x') + B\left[\mathbb{E}[x]\right] d\mu(x')\,d\mu(x'')$$

for some functionals $A$ and $B$, where

$$A[x] = \partial_{aa}\tilde{f}(x) - \frac{\int \partial_\lambda \tilde{f} d\mu}{\int \partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu}\left[\partial_{aa}g(x) + 2\partial_{ab}g(x)\,\partial_a b^*(x)\right]$$
$$- \frac{\int \partial_\lambda \tilde{f} d\mu}{\int \partial_b g(x)\,\partial_\lambda b^*(x)\,d\mu}\left[\partial_{bb}g(x)\,\partial_a b^*(x)^2 + \partial_b g(x)\,\partial_{aa}b^*(x)\right].$$

and

$$\partial_{aa}\tilde{f}(x) = \partial_{aa}f(x) + 2\partial_{ab}f(x)\,\partial_a b^*(x) + \partial_{bb}f(x)\left(\partial_a b^*(x)\right)^2 + \partial_b f(x)\,\partial_{aa}b^*(x)$$

After regrouping the terms, substituting $\partial_{aa}\tilde{f}$ and $\partial_\lambda \tilde{f}$, an application of the Dominated

Convergence Theorem implies that the limit is given by

$$\begin{aligned}A[\mathbb{E}[x]] = \partial_{aa} f(\mathbb{E}[x]) - \frac{\partial_b f(\mathbb{E}[x])}{\partial_b g(\mathbb{E}[x])}\partial_{aa} g(\mathbb{E}[x]) \\ + 2\left(\partial_{ab} f(\mathbb{E}[x]) - \frac{\partial_b f(\mathbb{E}[x])}{\partial_b g(\mathbb{E}[x])}\partial_{ab} g(\mathbb{E}[x])\right)\partial_a b^*(\mathbb{E}[x]) \\ + \left(\partial_{bb} f(\mathbb{E}[x]) - \frac{\partial_b f(\mathbb{E}[x])}{\partial_b g(\mathbb{E}[x])}\partial_{bb} g(\mathbb{E}[x])\right)(\partial_a b^*(\mathbb{E}[x]))^2 .\end{aligned}$$

Thus, we can rewrite

$$\begin{aligned}S_b[a,x] - S_b[a^*,x] &= \frac{1}{2}A[\mathbb{E}[x]]\int_\Omega\int_\Omega 1_{x'=x''}\Delta\hat{a}(x')\Delta\hat{a}(x'')\,d\mu(x') \\ &\quad + \frac{1}{2}B[\mathbb{E}[x]]\int_\Omega\int_\Omega \Delta\hat{a}(x')\Delta\hat{a}(x'')\,d\mu(x')\,d\mu(x'') + o\left(\|x-\mathbb{E}[x]\|^2\right) \\ &= \frac{1}{2}A[\mathbb{E}[x]]\int_\Omega \Delta\hat{a}(x')^2\,d\mu(x') + o\left(\|x-\mathbb{E}[x]\|^2\right) \\ &= \frac{1}{2}A[\mathbb{E}[x]]\,\mathbb{E}\left[\Delta a(x)^2\right] + o\left(\|x-\mathbb{E}[x]\|^2\right)\end{aligned}$$

where the latter terms grouped with $B$ drop out due to centering. The first-order conditions of this program along $a_{x_l}$ therefore characterize the second-order solution,

$$\begin{aligned}0 &= \mathbb{E}\left[\left\{a_0 + a_{x_l}^\intercal x_l - \left(a^*(\mathbb{E}[x]) + \partial_x a^*(\mathbb{E}[x])^\intercal(x-\mathbb{E}[x])\right)\right\}x_l\right] \\ &= \mathbb{E}\left[a_0 + a_{x_l}^\intercal x_l - \left(a^*(\mathbb{E}[x]) + \partial_x a^*(\mathbb{E}[x])^\intercal(x-\mathbb{E}[x])\right)\right].\end{aligned}$$

From the second equation, it follows that $a(\mathbb{E}[x]) = a^*(\mathbb{E}[x])$, verifying the initial conjecture. Thus, when $A[\mathbb{E}[x]] < 0$, the problem reduces to minimizing the tracking error (or maximizing it in the converse)

$$\begin{aligned}S_b[a] - S_b[a^*] &\propto -\mathbb{E}\left[\left\{a_{x_l}^\intercal(x_l - \mathbb{E}[x_l]) - \partial_x a^*(\mathbb{E}[x])^\intercal(x-\mathbb{E}[x])\right\}^2\right] \\ &= -a_{x_l}^\intercal \text{Var}(x_l)\,a_{x_l} - \partial_x a^*(\mathbb{E}[x])^\intercal \text{Var}(x)\,\partial_x a^*(\mathbb{E}[x]) + 2a_{x_l}^\intercal \text{Cov}(x_l,x)\,\partial_x a^*(\mathbb{E}[x])\end{aligned}$$

which has the optimal solution of

$$a_{x_l} = \text{Var}(x_l)^{-1}\,\text{Cov}(x_l,x)\,\partial_x a^*(\mathbb{E}[x]).$$

□

From this, the following Projection corollary follows.

**Corollary 2.** *Suppose the profit function satisfies $\partial_{PP}\pi \leq 0$, $\partial_{PQ}\pi \geq 1$, $\partial_{QQ}\pi \leq 0$.*

*Let $P^*(x)$ denote the flexible pricing solution. As $x \to \mathbb{E}[x]$, to a second-order approximation the seller's optimal currency choice is the unconditional passthrough of exchange rates onto flexible prices,*

$$\beta^* = \text{Var}\,(s)^{-1}\,\text{Cov}\,(s,x)\,\partial_x \bar{P}^*\left(\mathbb{E}\left[x\right]\right).$$

*Proof.* The incentive and participation constraints are analytical and the willingness to pay has the codomain of the positive real line, so Lagrange multipliers exist. Both a participation and ex post incentive constraint satisfy the representation of $G$ in Lemma 2. The control is also multilinear as $P(x) = P_0 + \beta \cdot s$.

What remains to be shown is that the second-order condition is satisfied, that is to say that $A\left(\mathbb{E}\left[x\right]\right) \leq 0$, where $A$ is defined in Lemma 3. Careful algebra reveals that

$$\begin{aligned} A\left(\mathbb{E}\left[x\right]\right) = M^i \partial_{PP}\pi + 2M^i \left(\partial_{PQ}\pi\left(\mathbb{E}\left[x\right]\right) - \frac{\partial_Q \pi\left(\mathbb{E}\left[x\right]\right)}{\bar{V} - \bar{P}}\right)\partial_P Q^*\left(\mathbb{E}\left[P\right], x\right) \\ + M^i \partial_{QQ}\pi\left(\mathbb{E}\left[x\right]\right)\left(\partial_P Q^*\left(\mathbb{E}\left[P\right], x\right)\right)^2 \end{aligned}$$

Note that the SDF is strictly positive. The first term is assumed to be negative. The second term follows by limited liability and the fact that $\partial_P Q \leq 0$. The third term is weakly negative by assumption. Consequently, $A\left(\mathbb{E}\left[x\right]\right) \leq 0$, completing the proof. □

### A.1.3 Proof of Proposition 2

*Proof.* The optimal flexible price is given by

$$\left(M^i\left(\partial_P\pi + \partial_Q\pi\partial_P Q\right) + M^j \mathbb{E}\left[M^i \partial_Q \pi \partial_{\mathbb{E}[M^j P]} Q\right]\right)\mu(x) = 0 \qquad \forall x \in X.$$

Linearizing this yields

$$\begin{aligned} o\left(\|x - \mathbb{E}\left[x\right]\|\right) = \bar{\pi}_P\left(M^i - \bar{M}^i\right) + \bar{M}^i \bar{\pi}_{PP}\partial_x \bar{P}^*\left(x - \bar{x}\right) \\ + \bar{M}^i \bar{\pi}_{Px}\left(x - \bar{x}\right) + \bar{M}^i \partial_Q \bar{\pi} \partial_{\mathbb{E}[M^j P]}\bar{Q}\left(M^j - \bar{M}^j\right) + \text{constants.} \end{aligned}$$

Using the integrated FOC $\bar{M}^i \bar{\pi}_P = -\bar{M}^j \bar{M}^i \partial_Q \bar{\pi} \partial_{\mathbb{E}[M^j P]}\bar{Q}$,

$$\partial_x \bar{P}^*\left(x - \bar{x}\right) \approx -\left(\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}}\left(x - \bar{x}\right) + \frac{\bar{\pi}_P}{\bar{\pi}_{PP}}\left(\frac{M^i}{\bar{M}^i} - \frac{M^j}{\bar{M}^j}\right)\right) + \text{constants.}$$

It can be verified that $\bar{\pi}_P = \delta \partial_P \bar{\pi}$ in the seller's problem. Substitute in Definition 2. Applying Corollary 2 completes the proof. $\square$

### A.1.4 Proof of Proposition 3

**Lemma 4.** *Let $x \to \mathbb{E}[x]$. The optimal currency choice minimizes the tracking error to the optimal FX passthrough*

$$\arg \min_{\beta \in \{0, e_1, \ldots, e_n\}} (\beta - \beta^*)^\intercal \, \text{Var}\,(s)\,(\beta - \beta^*)\,.$$

*Proof.* Since $P_0$ is a control, we know that for any choice of $\beta$, the expected price must be centered. Thus, the objective is proportional to the tracking error of the flexible pricing solution $P^*(x)$.

$$\begin{aligned}
&\mathbb{E}\left[\left\{\beta^\intercal (s - \mathbb{E}[s]) - \left(\partial_x \left(\bar{P}^*(\mathbb{E}[x])\right)^\intercal (x - \mathbb{E}[x])\right)\right\}^2\right] \\
=&\mathbb{E}\left[\left\{(\beta - \beta^* + \beta^*)^\intercal (s - \mathbb{E}[s]) - \left(\partial_x \left(\bar{P}^*(\mathbb{E}[x])\right)^\intercal (x - \mathbb{E}[x])\right)\right\}^2\right] \\
=&(\beta - \beta^* + \beta^*)^\intercal \, \text{Var}\,(s)\,(\beta - \beta^* + \beta^*) + \partial_x \bar{P}^*(\mathbb{E}[x])^\intercal \, \text{Var}\,(x)\, \partial_x \bar{P}^*(\mathbb{E}[x]) \\
&\qquad - 2\,(\beta - \beta^* + \beta^*)\, \text{Cov}\,(s, x)\, \partial_x \bar{P}^*(\mathbb{E}[x]) \\
=&(\beta - \beta^*)^\intercal \, \text{Var}\,(s)\,(\beta - \beta^*) + 2\,(\beta - \beta^*)^\intercal \, \text{Var}\,(s)\,(\beta^*) + (\beta^*)^\intercal \, \text{Var}\,(s)\,\beta^* \\
&\qquad - 2\,(\beta - \beta^*)^\intercal \, \text{Cov}\,(s, x)\, \partial_x \bar{P}^*(\mathbb{E}[x]) - 2\,(\beta^*)^\intercal \, \text{Cov}\,(s, x)\, \partial_x \bar{P}^*(\mathbb{E}[x])
\end{aligned}$$

since $\beta$ is the control, we can rewrite this as proportional to

$$\propto (\beta - \beta^*)^\intercal \left[\text{Var}\,(s)\,(\beta - \beta^*) + 2\text{Var}\,(s)\,\beta^* - 2\text{Cov}\,(s, x)\, \partial_x \bar{P}^*(\mathbb{E}[x])\right]$$

using the condition

$$\beta^* = \text{Var}\,(s)^{-1}\, \text{Cov}\,(s, x)\, \partial_x \bar{P}^*(\mathbb{E}[x])$$

we get

$$\propto (\beta - \beta^*)^\intercal \, \text{Var}\,(s)\,(\beta - \beta^*)\,.$$

$\square$

From this, the binary choice solution is immediate in Proposition 3.

**Corollary 3.** *For the PCP $\beta/P_0 = 0$ vs LCP $\beta/P_0 = 1$ discrete choice problem, to a* *second-order approximation, the threshold rule is*

$$\beta = \begin{cases} 0 & \beta^*_\delta / P_0 < 1/2 \\ 1 & o.w. \end{cases}, \qquad as\ x \to \mathbb{E}[x]$$

**Corollary 4.** *For the $n$ currency discrete choice problem, if the covariance matrix is a diagonal matrix with constant variance, the threshold rule minimizes the Euclidean distance.*

### A.1.5 Equivalence to Forward Contracts

While the 1/2 threshold rule applies to the case where $\delta < 1$, it fails when quantities are entirely fixed $\delta = 1$.[21] In this context, I formalize an equivalence between currency choice in goods trade and in the forward exchange rate hedging problem.

I define the forward pricing problem as a setting in which a market participant (seller $i$) writes a forward contract with a market maker (buyer $j$), locking in a future exchange rate $\hat{F}$ for the transfer $\beta S'(\omega)$. The proposed forward price must at least meet the reservation value of the market maker, giving rise to a participation constraint. And as before, the seller $i$ exercises all the bargaining power.

**Definition 8.** The **forward pricing problem** is a currency denomination $\beta \in [0, 1]$ and forward price $F \in \mathbb{R}_{++}$ that maximizes a market participant's ex ante surplus, subject to market maker participation constraint:

$$\begin{aligned} (\beta, F) &\in \arg\max_{\hat{\beta}, \hat{F}} \mathbb{E}\left[M^i\left(\hat{\beta}S' - \hat{F}\right)\right] \\ \text{s.t. } 0 &\leq \mathbb{E}\left[M^j\left(\hat{F} - \hat{\beta}S'\right)\right]. \end{aligned}$$

The forward pricing problem exists for the sole purpose of financial hedging. Currencies do not facilitate real trade in this setting since ex post transfers net out to zero

$$\underbrace{\hat{\beta}S'(\omega) - \hat{F}_0}_{\text{Mkt Participant Transfers}} + \underbrace{\hat{F}_0 - \hat{\beta}S'(\omega)}_{\text{Mkt Maker Transfers}} = 0 \qquad \forall \omega \in \Omega.$$

Nonetheless, the market participant captures ex ante surplus $\mathcal{S}$. Normalizing the participation constraint by $\mathbb{E}[M^j]$ and the participant's objective by $\mathbb{E}[M^i]$, one can net out the

[21] The flexible pricing solution becomes bang-bang, violating both the second-order condition and the continuity assumption.

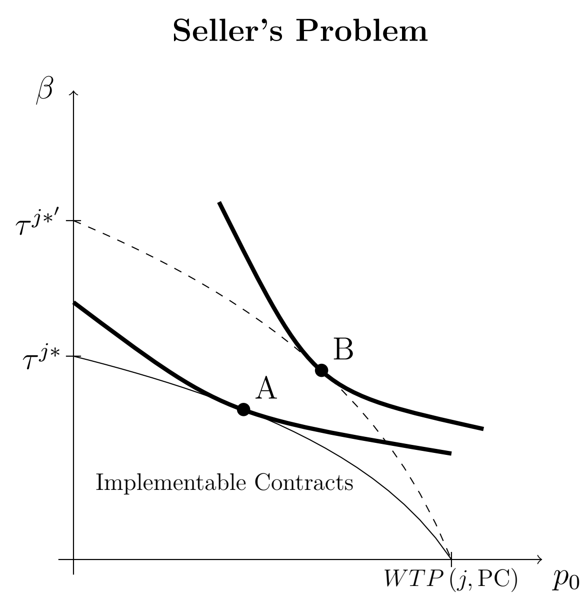


Figure 5: An increase in the buyer's local currency preference $\tau^{j*} \to \tau^{j*'}$ creates room for risk sharing. In the seller's problem, this changes currency denomination $\beta$ from $A \to B$ and for forward pricing from $C \to D$.

transfer to demonstrate that the surplus is proportional to the total quantity of risk sharing

$$\mathcal{S} \propto \hat{\beta}\mathbb{E}\left[\left(\frac{M^i}{\mathbb{E}\left[M^i\right]} - \frac{M^j}{\mathbb{E}\left[M^j\right]}\right) S'\right] = \hat{\beta}\Delta_{ij}\tau.$$

In the forward pricing problem, optimal currency choice $\hat{\beta} \in [0, 1]$ is a corner solution that maximizes the surplus extracted from the relative cost of FX hedging $\Delta_{ij}\tau$. This stands in contrast to the real hedging theory, which features an interior currency choice that minimizes the tracking error between the realized and flexible monopoly price.

**Lemma 5.** *In the forward pricing problem, the optimal currency denomination satisfies*

$$\beta = \begin{cases} 0 & \Delta_{ij}\tau < 0 \\ 1 & \Delta_{ij}\tau \geq 0 \end{cases}.$$

The visual intuition for the seller's problem and forward pricing problem is characterized by Figure 5. In this figure, a seller chooses between producer and local currency pricing. The set of implementable price-quantity pairs is given by the region under the curve, with the x-axis representing the buyer's willingness to pay in producer terms and the y-axis representing the currency denomination scalar $\beta$. The bolded lines represent ex ante seller surplus curves.

In the seller's problem the optimal currency share is given by $A$. $A$ is on the frontier of implementable contracts satisfying the participation constraint. An increase in the seller's preference for the local currency shifts the implementable contract region upwards, leading to an increased local currency share given by $B$. The response is tempered by the real hedging incentive, reflected in convex shape of the seller's surplus curves in bold. In the forward pricing problem, the real hedging consideration disappears. The response becomes dramatic causing a total shift in the currency share from $C$ to $D$.

The full contracting problem becomes identical to the forward pricing problem when quantities become fixed.

**Proposition 12.** *Let profits satisfy $\partial_P \pi = Q$. The optimal currency choice of the fixed quantity problem $\delta = 1$ is equivalent to that of the forward pricing solution, as in Lemma 5.*

*Proof.* The seller's problem is given by

$$\max \mathbb{E}\left[M^i \pi\left(P, Q, x\right)\right] \qquad \text{s.t. } Q = \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right).$$

The perturbation of $\beta$ yields the effect

$$\begin{aligned} &\mathbb{E}\left[M^i\left(\partial_P \pi s + \partial_Q \pi \partial_{\mathbb{E}[M^j P]} \bar{V}^{-1} \mathbb{E}\left[M^j s\right]\right)\right] \\ =&\mathbb{E}\left[M^i Q s\right] + \mathbb{E}\left[M^i \partial_Q \pi \partial_{\mathbb{E}[M^j P]} \bar{V}^{-1}\right] \mathbb{E}\left[M^j s\right] \end{aligned}$$

using the fact that the perturbation of $P_0$ implies

$$\mathbb{E}\left[M^i Q\right] + \mathbb{E}\left[M^i \partial_Q \pi \partial_{\mathbb{E}[M^j P]} \bar{V}^{-1}\right] \mathbb{E}\left[M^j\right] = 0$$

we can rewrite the effect of the perturbation as

$$= \mathbb{E}\left[M^i s\right] - \frac{\mathbb{E}\left[M^i\right]}{\mathbb{E}\left[M^j\right]} \mathbb{E}\left[M^j s\right] \propto \Delta_{ij} \tau.$$

□

## A.2 Details of the Simple Example

Assume the profit function is

$$\pi\left(P, Q, x\right) = PQ - CQ$$

and the demand curves satisfy

$$V\left(Q, x\right) = \mathcal{P}\left(Q / \mathcal{Q}\right)^{-1/\sigma}.$$

$V$ satisfies two identities. Two relationships must hold

$$V^{-1}(P,x) = \mathcal{Q}(P/\mathcal{P})^{-\sigma}; \qquad \Rightarrow \partial_P V^{-1}(P,x) = -\sigma \frac{V^{-1}(P,x)}{P}$$

and second, using the inverse function theorem,

$$\partial_{\mathbb{E}[M^j P]} \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right) = \frac{1}{\mathbb{E}\left[M^j V_Q\left(\bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right), x\right)\right]}.$$

To characterize optimal currency choice, we start by characterizing the flexible price. The flexible price corresponds to the expected price of the nominal rigid solution. The flexible price satisfies the first-order condition

$$M^i\left(\partial_P \pi + \partial_Q \pi \partial_P Q\right) + M^j \mathbb{E}\left[M^i \partial_Q \pi \partial_{\mathbb{E}[M^j P]} Q\right] = 0.$$

Using the fact that

$$Q = (1-\delta) V^{-1}(P,x) + \delta \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right)$$

we can substitute in with

$$M^i\left(Q - \sigma (P-C)(1-\delta)\frac{V^{-1}(P,x)}{P}\right) - \sigma \frac{\delta M^j \mathbb{E}\left[M^i (P-C)\right]}{\mathbb{E}\left[M^j \frac{V\left(\bar{V}^{-1}(\mathbb{E}[M^j P]),x\right)}{\bar{V}^{-1}(\mathbb{E}[M^j P])}\right]} = 0.$$

Taking the deterministic steady state $x(\omega) \to \bar{x}$, we get

$$\bar{M}^i\left(\bar{Q} - \sigma\left(1 - \bar{C}/\bar{P}\right)(1-\delta)\bar{Q}^m\right) - \sigma\delta \frac{\bar{M}^j \bar{M}^i\left(\bar{P} - \bar{C}\right)}{\bar{M}^j \bar{P}} \bar{Q}_0 = 0$$

using the fact that $\lim_{x\to\bar{x}} V^{-1}(P,x) = \lim_{x\to\bar{x}} \bar{V}^{-1}\left(\mathbb{E}\left[M^j P\right]\right) = \bar{Q}$, we get

$$0 = \bar{Q} - \sigma\left(1 - \bar{C}/\bar{P}\right)\bar{Q}$$

implying

$$\bar{P} = \sigma\left(\bar{P} - \bar{C}\right) \Rightarrow \bar{P}^* = \frac{\sigma}{\sigma - 1}\bar{C}.$$

Given the standard solution, we now characterize the coefficients $\bar{\pi}_P$, $\bar{\pi}_{PP}$, and $\bar{\pi}_{Px}$ as these show up in the currency denomination formula. We have that

$$\begin{aligned}\bar{\pi}_P &= \bar{Q} - \sigma\left(1 - \bar{C}/\bar{P}\right)(1-\delta)\bar{Q} \\ &= \bar{Q}\left(1 - \sigma\left(1 - \bar{C}/\bar{P}\right)\right) + \delta\sigma\left(1 - \bar{C}/\bar{P}\right)\bar{Q} = 0\end{aligned}$$

using the first order condition we can rewrite

$$\bar{\pi}_P = -\delta\sigma\left(1 - \bar{C}/\bar{P}\right)\bar{Q} = \delta\bar{Q}.$$

For the second-order condition, we get

$$\begin{aligned}\pi_{PP} &= \partial_P \pi_P + \partial_Q \pi_P \partial_P Q \\ &= -\sigma\frac{C}{P^2}\left(1-\delta\right)Q^m + \partial_P Q\left(1 - \sigma\left(1 - C/P\right)\left(1-\delta\right)\right)\end{aligned}$$

which at the deterministic steady state is

$$\begin{aligned}\bar{\pi}_{PP} &= -\sigma\bar{C}/\bar{P}^2\left(1-\delta\right)\bar{Q} - \sigma\left(1-\delta\right)\bar{Q}/\bar{P}\left(1-\sigma\left(1-\bar{C}/\bar{P}\right)\left(1-\delta\right)\right) \\ &= -\sigma\bar{C}/\bar{P}\left(1-\delta\right)\bar{Q}/\bar{P} - \sigma\bar{Q}/\bar{P}\delta\left(1-\delta\right) \\ &= -\sigma\bar{Q}/\bar{P}\left(1-\delta\right)\left(\bar{C}/\bar{P}+\delta\right) \\ &= -\sigma\bar{Q}/\bar{P}\left(1-\delta\right)\left(\frac{\sigma-1}{\sigma}+\delta\right).\end{aligned}$$

Moreover,

$$\begin{aligned}\pi_{Px} &= \partial_x \pi_P + \partial_Q \pi_P \partial_x Q \\ &= \partial_x Q\left(1-\sigma\left(1-C/P\right)\left(1-\delta\right)\right) + \sigma\partial_x C/P\left(1-\delta\right)Q^m\end{aligned}$$

and so at the limit

$$\bar{\pi}_{Px} = \partial_x\bar{Q}\delta + \sigma\partial_x\bar{C}\left(1-\delta\right)\bar{Q}/\bar{P}.$$

Assembling the pieces, we get

$$\begin{aligned}-\frac{\bar{\pi}_{Px}}{\bar{\pi}_{PP}} &= \frac{\partial_x Q\delta}{\sigma\bar{Q}/\bar{P}\left(1-\delta\right)\left(\frac{\sigma-1}{\sigma}+\delta\right)} + \frac{\sigma\partial_x\bar{C}\left(1-\delta\right)\bar{Q}/\bar{P}}{\sigma\bar{Q}/\bar{P}\left(1-\delta\right)\left(\frac{\sigma-1}{\sigma}+\delta\right)} \\ &= \frac{\delta\bar{P}\partial_x q}{\left(1-\delta\right)\left(\sigma-1+\delta\sigma\right)} + \frac{\frac{\sigma-1}{\sigma}\partial_x\bar{C}/\bar{C} * \bar{P}}{\frac{\sigma-1}{\sigma}+\delta}\end{aligned}$$

and

$$-\frac{\bar{\pi}_P}{\bar{\pi}_{PP}} = \frac{\delta\bar{Q}}{\sigma\bar{Q}/\bar{P}\left(1-\delta\right)\left(\frac{\sigma-1}{\sigma}+\delta\right)} = \frac{\delta}{\left(1-\delta\right)\left(\sigma-1+\delta\sigma\right)}\bar{P}$$

thus

$$\beta^*/\bar{P} \approx \frac{\delta\partial_x\bar{q}b_{xs} + \delta\frac{\Delta_{ij}\tau}{\text{Var}\left(s\right)}}{\left(1-\delta\right)\left(\sigma-1+\delta\sigma\right)} + \frac{\frac{\sigma-1}{\sigma}}{\frac{\sigma-1}{\sigma}+\delta}\frac{\partial_x\bar{C}}{\bar{C}}b_{xs}.$$

Because the willingness to pay index is uncorrelated to the exchange rate $\partial_x \bar{q} b_{xs} = 0$, which reduces this down to

$$\beta^* / \bar{P} \approx \frac{\delta}{\text{Var}\,(s)\,(1-\delta)\,(\sigma - 1 + \delta\sigma)} \Delta_{ij} \tau + \frac{\frac{\sigma-1}{\sigma}}{\frac{\sigma-1}{\sigma} + \delta} \frac{\partial_x \bar{C}}{\bar{C}} b_{xs}.$$

Combining with the identity $\mu = \frac{\sigma-1}{\sigma}$ concludes the proof.

## A.3 Proof of Proposition 4

The problem is set up as

$$\begin{aligned}\mathcal{L} = \min_{\lambda \in C(X), \eta \geq 0} \max_{P,Q \in C(X)^2} & \int M^i \pi\,(P, Q, x) + \eta M^j v\,(P, Q, x)\, d\mu \\ & + \int \lambda\,(x)\,[v\,(P, Q, x) - \tilde{v}\,(P, x)]\, d\mu\end{aligned}$$

where the set of implementable deviations is reduced down to quantities by the nominal rigidity assumption $\tilde{v}\,(P\,(x)\,, x) = \arg\max_{\hat{Q} \in \mathbb{D}^{rng}_{jQ}(x)} v\left(P\,(x)\,, \hat{Q}, x\right)$. A set of Lagrange multipliers exist since the differential operator is defined for $v$ and an optimal contract is assumed to exist.

Thus, we may take the first-order conditions of this with respect to $Q\,(x)$ as

$$M^i \partial_Q \pi\,(x) + \eta M^j \partial_Q v\,(x) + \lambda\,(x)\, \partial_Q v\,(x) = 0.$$

When $\lambda\,(x) > 0$, it follows that $v\,(P\,(x)\,, Q\,(x)\,, x) = \tilde{v}\,(P\,(x)\,, x)$. By the implicit function theorem, define $Q^{rng}\,(P\,(x)\,, x)$ to describe this relationship and note that it is of class $C^\infty$.

Consequently, piece-wise we may express

$$Q^*\,(x) = \begin{cases} Q\,(P\,(x)\,, x, \eta) & \lambda\,(x) = 0 \\ Q^{rng}\,(P\,(x)\,, x) & \lambda\,(x) > 0 \end{cases}$$

where $\eta$ is the scalar Lagrange multiplier on the participation constraint. The set of states $X^{cmt} := \{x \in X : v\,(P\,(x)\,, Q\,(x)\,, x) > \tilde{v}\,(P\,(x)\,, x)\}$ can therefore be expressed as $X^{cmt} = \{x \in X : \lambda\,(x) = 0\}$. The forward direction of the proof is therefore immediate: if $\mu\,(X^{cmt}) = 0$, quantities are flexible.

To finish the proof, we show the reverse. Suppose for contradiction that $\mu\,(X^{cmt}) > 0$ and quantities are flexible. There is a positive measure of states such that $Q$ is described by the implicit relationship

$$M^i \partial_Q \pi\,(x) + \eta M^j \partial_Q v\,(x) = 0.$$

Since $\partial_Q \pi(x) > 0$ and $M^i > 0$ is a stochastic discount factor, it follows that $\eta M^j \partial_Q v(x) < 0$. Consequently $\eta > 0$ as $M^j > 0$ and $\partial_Q v(x) < 0$.

Thus the participation constraint holds for some neighborhood around $Q^*$ and $\eta^*$.

$$\mathbb{E}\left[M^j \left[1_{\lambda=0} v\left(P, Q\left(P, x, \eta^*\right), x\right) + \left(1 - 1_{\lambda=0}\right) \tilde{v}\left(P, x\right)\right]\right] = 0$$

the implicit function as applied to Banach spaces allows us to rewrite $\eta : C(X) \times C(\Omega) \mapsto \mathbb{R}_+$

$$\eta^* = \eta\left[P, x\right].$$

Consequently, $Q$ is a functional of $P$ and $x$, that is to say

$$Q^*(x) = \begin{cases} Q\left(P(x), x, \eta\left[P, x\right]\right) & \lambda(x) = 0 \\ Q^{rng}\left(P(x), x\right) & \lambda(x) > 0 \end{cases}.$$

Because $\lambda(x) = 0$ happens with positive probability, it follows by contradiction that flexible quantities imply $\mu\left(X^{cmt}\right) = 0$.

## A.4 Proof of Proposition 5

Since $Q^*$ satisfies the IC and PC, we may substitute out the constraints and directly maximize

$$\max_{P \in C(X)} \int_{\lambda=0} M^i \pi\left(P, Q\left(P, x, \eta\left[P, x\right]\right), x\right) d\mu + \int_{\lambda>0} M^i \pi\left(P, Q^{rng}\left(P, x\right), x\right).$$

Pointwise first order conditions for $P$ are given by

$$M^i\left[\partial_P \pi + \partial_Q \pi \partial_P Q\right] + \mathbb{E}\left[M^i \partial_Q \pi \partial_\eta Q 1_{\lambda=0}\right] \frac{\delta \eta}{\delta P} 1_{\eta>0} = 0$$

when $\lambda(x) = 0$. Otherwise, when differentiating by $P(x)$ for $\lambda(x) > 0$, we get

$$M^i\left[\partial_P \pi + \partial_Q \pi \partial_P Q^{rng}\right] 1_{\lambda>0} = 0.$$

The participation constraint meanwhile reads as

$$\eta\left[\int_{\lambda=0} M^j v\left(P(x), Q(x), x\right) d\mu + \int_{\lambda>0} M^j \tilde{v}\left(P(x), x\right)\right] = 0.$$

To recover $\delta\eta/\delta P 1_{\eta>0}$, totally differentiate the constraint with $\lambda(x) = 0$,

$$M^j [\partial_P v + \partial_Q v \partial_P Q] \int 1_x d\mu(x) + \mathbb{E}\left[M^j 1_{\lambda=0} \partial_Q v \partial_\eta Q\right] \frac{\delta\eta}{\delta P} = 0$$

thus

$$\frac{\delta\eta}{\delta P} = -M^j \frac{[\partial_P v + \partial_Q v \partial_P Q]}{\mathbb{E}[M^j 1_{\lambda=0} \partial_Q v \partial_\eta Q]} \int 1_x d\mu(x)$$

and so we may rewrite

$$M^i \pi_P - M^j [\partial_P v + \partial_Q v \partial_P Q] \frac{\mathbb{E}[M^i 1_{\lambda=0} \partial_Q \pi \partial_\eta Q]}{\mathbb{E}[M^j 1_{\lambda=0} \partial_Q v \partial_\eta Q]} 1_{\eta>0} = 0.$$

Note that $\partial_P v + \partial_Q v \partial_P Q$ is the financial risk on $v$ as defined in Definition 1. Thus

$$M^i \pi_P - M^j v_P \frac{\mathbb{E}[M^i 1_{\lambda=0} \partial_Q \pi \partial_\eta Q]}{\mathbb{E}[M^j 1_{\lambda=0} \partial_Q v \partial_\eta Q]} 1_{\eta>0} = 0.$$

Because $\mathbb{D}_j^{rng}$ is a continuous correspondence, the function $\tilde{v}(P(x), x)$ is smooth respect to $x$. This allows us to apply Corollary 2 to analyze $\beta^*$. Taking $x \to \mathbb{E}[x]$,

$$\lim_{x \to \mathbb{E}[x]} \frac{\mathbb{E}[M^i 1_{\lambda=0} \partial_Q \pi \partial_\eta Q]}{\mathbb{E}[M^j 1_{\lambda=0} \partial_Q v \partial_\eta Q]} = \frac{\bar{M}^i \partial_Q \bar{\pi} \partial_\eta \bar{Q}}{\bar{M}^j \partial_Q \bar{v} \partial_\eta \bar{Q}} = \frac{\bar{M}^i \partial_Q \bar{\pi}}{\bar{M}^j \partial_Q \bar{v}}.$$

Linearizing this first-order condition, we get

$$\begin{aligned} o(\|x - \mathbb{E}[x]\|) = & \left(M^i - \bar{M}^i\right) \bar{\pi}_P + \bar{M}^i \bar{\pi}_{PP} \partial_x \bar{P}(x - \bar{x}) + \bar{M}^i \bar{\pi}_{Px}(x - \bar{x}) \\ & - \left(M^j - \bar{M}^j\right) \bar{v}_P \frac{\bar{M}^i \partial_Q \bar{\pi}}{\bar{M}^j \partial_Q \bar{v}} 1_{\eta>0} - \frac{\bar{M}^i \partial_Q \bar{\pi}}{\partial_Q \bar{v}} \bar{v}_{PP} 1_{\eta>0} \partial_x \bar{P}(x - \bar{x}) \\ & - \frac{\bar{M}^i \partial_Q \bar{\pi}}{\partial_Q \bar{v}} \bar{v}_{Px} 1_{\eta>0}(x - \bar{x}) \end{aligned}$$

with the additional restriction $\bar{\pi}_P = \bar{v}_P \frac{\partial_Q \bar{\pi}}{\partial_Q \bar{v}} 1_{\eta>0}$, this becomes

$$\begin{aligned} o(\|x - \mathbb{E}[x]\|) = & \left(\frac{M^i}{\bar{M}^i} - \frac{M^j}{\bar{M}^j}\right) \bar{\pi}_P + \left(\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P} \bar{v}_{PP}\right) \partial_x \bar{P}(x - \bar{x}) \\ & + \left(\bar{\pi}_{Px} - \frac{\bar{\pi}_P}{\bar{v}_P} \bar{v}_{Px}\right)(x - \bar{x}). \end{aligned}$$

By projection, it follows that

$$\beta^* \approx -\left(\underbrace{\frac{\bar{\pi}_{Px} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{Px}}{\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}} b_{xs}}_{\text{Real Hedging}} + \underbrace{\bar{\pi}_P \frac{\mu\left(X^{cmt}\right)\Sigma^{-1}\Delta_{ij|X^{cmt}\mathcal{T}}}{\bar{\pi}_{PP} - \frac{\bar{\pi}_P}{\bar{v}_P}\bar{v}_{PP}}}_{\text{Financial Hedging}}\right).$$

## A.5 Proof of Proposition 7

For this proof, we proceed in two steps. First, we characterize the social planner's solution without nominal rigidities. It can be verified that the social planner's problem, after substituting in the home currency bond, labor, and consumption constraints is subject to a nontradables and trade balance constraint:

$$\begin{aligned}
\max_{\{C_{NT,t}, C_{T,t}, L_{T,t}, L_{NT,t}, B_{F,t}\}} & \sum_{t=0}^{1} \mathbb{E}_0 \left[U\left(C_{NT,t}, C_{T,t}, L_{T,t} + L_{NT,t}\right)\right] \\
\text{s.t. } & P^*_{T,t} C_{T,t} + S_t B_{F,t} \leq P_{T,t} A_{T,t} L_{T,t} + S_t R^*_{t-1} B_{F,t-1} \\
& A_{NT,t} L_{NT,t} \geq C_{NT,t}
\end{aligned}$$

Denote the associated multipliers $\mu^{NT}$ and $\mu^{TB}$. The first-order conditions are characterized by

$$\frac{\partial_L U_t}{\partial_{C_{NT}} U_t} = -A_{NT,t}$$

as the consumption-leisure tradeoff,

$$\frac{\partial_{C_T} U_t}{\partial_{C_{NT}} U_t} = \frac{P^*_{T,t}}{\mu_t^{NT}/\mu_t^{TB}}$$

as the expenditure switching mechanism

$$\frac{P_{T,t} A_{T,t}}{A_{NT,t}} = \frac{\mu_t^{NT}}{\mu_t^{TB}}$$

the labor allocation margin

$$\begin{aligned}
\mu_t^{TB} S_t &= \mathbb{E}_t\left[\mu_{t+1}^{TB} S_{t+1} R_t^*\right] \\
\iff 1 &= \mathbb{E}_t\left[\beta \frac{\partial_{C_T} U_{t+1}}{\partial_{C_T} U_t} \frac{P^*_{T,t}}{P^*_{T,t+1}} \frac{S_{t+1}}{S_t} R_t^*\right]
\end{aligned}$$

and the international risk sharing condition. These are the four conditions of efficiency.

Now, let us characterize the private solution. The household first-order conditions are

$$-\frac{\partial_L U_t}{\partial_{C_{NT}} U_t} = \frac{W_t}{P_{NT,t}}$$

as the consumption-leisure tradeoff,

$$\frac{\partial_{C_T} U_t}{\partial_{C_{NT}} U_t} = \frac{P^*_{T,t}}{P_{NT,t}}$$

as the T-NT tradeoff,

$$\begin{aligned} 1+\tau^{B^H} &= \mathbb{E}_t\left[\beta \frac{\partial_{C_T} U_{t+1}}{\partial_{C_T} U_t} \frac{P^*_{T,t}}{P^*_{T,t+1}} R_t\right] \\ 1+\tau^{B^F} &= \mathbb{E}_t\left[\beta \frac{\partial_{C_T} U_{t+1}}{\partial_{C_T} U_t} \frac{P^*_{T,t}}{P^*_{T,t+1}} \frac{S_{t+1}}{S_t} R^*_t\right] \end{aligned}$$

as the Euler Equations. Given the nontradable pricing solution $P_{NT} = \frac{W}{A_{NT}}$, the labor-leisure and expenditure-switching margins are efficient. To match the intertemporal risk sharing condition any first-best equilibrium must satisfy $\tau^{B^F} = 0$.

What is left is the firms labor demand, pricing decision, and currency of invoice. Following the results from the section A.2, the optimal price is given by

$$P_{T,t} + \beta \mathbb{E}_t\left[s_{t+1}\right] = \frac{\sigma}{\sigma-1}\left(1+\tau_L\right) \mathbb{E}_t\left[\frac{W_{t+1}}{A_{T,t+1}}\right]$$

with the associated optimal passthrough

$$\beta^*_\delta / P_{T,t} = \frac{\frac{\sigma-1}{\sigma}}{\frac{\sigma-1}{\sigma}+\delta} \frac{\operatorname{Cov}_t\left(W_{t+1}/A_{T,t+1}, s_{t+1}\right)}{\operatorname{Var}_t\left(s_{t+1}\right)} + \frac{\delta \partial_x \bar{q} b_{xs} + \delta \frac{\Delta_{ij}\tau}{\operatorname{Var}(s)}}{(1-\delta)(\sigma-1+\delta\sigma)}$$

because each exchange rate is independently distributed, it follows that $b_{xs} = 0$. Thus, defining $b_\tau := \frac{\delta}{(1-\delta)(\sigma-1+\delta\sigma)\operatorname{Var}(s)}$ and $\gamma := \frac{\frac{\sigma-1}{\sigma}}{\frac{\sigma-1}{\sigma}+\delta} \frac{\operatorname{Cov}_t\left(W_{t+1}/A_{T,t+1}, s_{t+1}\right)}{\operatorname{Var}_t(s_{t+1})}$ we have

$$\beta / P_{T,t} = \begin{cases} 0 & \frac{1+\tau_i^{B^F}}{1+\tau_i^{B^H}} < \frac{1}{b_\tau}\left(\frac{1}{2}-\gamma\right) + \frac{1+\tau_j^{B^F}}{1+\tau_j^{B^H}} \\ 1 & o.w. \end{cases}.$$

Finally, the labor allocation is given by

$$L_{T,t+1} = \frac{Q_{T,t+1}}{A_{T,t+1}}$$

where $Q_{T,t+1}$ is the demand given the realized price $P_{T,t}+\beta s_{t+1}$ for the demand curve specified in the seller's problem.

All that is left is to verify that the home currency bond tax, monetary policy, and labor subsidies recover the optimal labor allocation under PCP, and that PCP is privately optimal for the tradables sector. The labor allocation is optimal if and only if $Q_{T,t+1}$ achieves first best. This occurs when $P_{T,t} + \beta s_{t+1}$ equals the terms of trade implied under first-best,

$$P_{T,t} + \beta s_{t+1} = \frac{W_{t+1}}{A_{T,t+1}}.$$

Combining the labor subsidy $1+\tau_L = \frac{\sigma-1}{\sigma}$ with the firm's privately optimal price, one arrives at

$$P_{T,t} + \beta \mathbb{E}_t \left[s_{t+1}\right] = \mathbb{E}_t \left[\frac{W_{t+1}}{A_{T,t+1}}\right].$$

In conjunction with monetary policy $P_{NT,t+1} = \frac{A_{T,t+1}}{A_{NT,t+1}}$ and the competitive labor condition $P_{NT,t+1} = \frac{W_{t+1}}{A_{NT,t+1}}$ this becomes

$$P_{T,t} + \beta \mathbb{E}_t \left[s_{t+1}\right] = 1 = \frac{W_{t+1}}{A_{T,t+1}}.$$

Because exchange rates $s_{t+1}$ are stochastic, this condition holds for all states in $t+1$ iff $\beta = 0$. Thus, for $\beta = 0$ in conjunction with the first-best restriction $\tau^{B^F} = 0$, one arrives at

$$\frac{1}{1+\tau_i^{B^F}} < \frac{1}{2b_\tau} + \frac{1}{1+\tau_j^{B^H}}$$

because $\gamma = 0$ when marginal costs are constant.

## A.6 Proof of Proposition 8

To determine the Euler equation wedges, we now use the Arrow-Pratt portfolio choice method. Let the buyer's next period utility be summarized by the indirect utility function $U(W, x)$. The optimal wealth share of currencies today $\theta \in \mathbb{R}^n$ must satisfy

$$\max_\theta \mathbb{E}\left[U(W,x)\right]$$
$$\text{s.t. } W \leq W_0 \left[R + \theta^\intercal \left(R^* S'/S - R\right)\right]$$

It is well-known from the Arrow-Pratt solution technique that $\theta^\intercal = RRA_{ss}^{-1}\Sigma^{-1}\mathbb{E}\left[R^*S'/S - R\right]$ (Arrow, 1971; Pratt, 1964). However, as an exercise of validating Lemma 3, I prove it with the projection technique.

Begin by assuming the technical conditions hold ($x$ and $\theta$ are real, continuous, and bounded). Denote $q$ as a set of hypothetical Arrow-Debreu security prices which span the filtration $\mathcal{F}^x$. The trading strategy of purchasing an Arrow-Debreu security yields the rate of return $1(x) - Rq_x$ where $1(x)$ is the step function. With the fully indexed solution

$$W \leq W_0 \left[R + \zeta^\intercal \left(1(x) - Rq_x\right)\right]$$

the first-order conditions are given by the point-wise condition

$$U' W_0 \left(1(x) - Rq_x\right)^\intercal \mu(x) = 0.$$

Taking $x \to \mathbb{E}[x]$ and linearize this condition. Note that as a consequence $1(x) \to Rq_x$. In this limit,

$$\bar{U}'' W_0^2 \zeta^\intercal \left(1(x) - Rq_x\right)\left(1(x) - Rq_x\right)^\intercal + \bar{U}' W_0 \left(1(x) - Rq_x\right)^\intercal = 0.$$

rearrange this as

$$\zeta^\intercal = -\frac{\bar{U}'}{\bar{U}'' \cdot W_0} \left[\left(1(x) - Rq_x\right)\left(1(x) - Rq_x\right)^\intercal\right]^{-1} \left(1(x) - Rq_x\right)^\intercal.$$

Consequently, the buyer's optimal wealth in each state is

$$W = W_0 \left[R + \zeta^\intercal \left(1(x) - Rq_x\right)\right].$$

In the exchange-rate indexed solution, the buyer's optimal wealth in each state is instead

$$W = W_0 \left[R + \theta^\intercal \left(R^* S'/S - R\right)\right].$$

Thus, the projection method implies that

$$\begin{aligned} \theta^\intercal &\approx \text{Var}\left(R^* S'/S - R\right)^{-1} \text{Cov}\left(R^* S'/S - R, 1_x\right) \zeta^\intercal \\ &= R * RRA^{-1} \text{Var}\left(R^* S'/S - R\right)^{-1} \text{Cov}\left(R^* S'/S - R, 1(x)\right) \\ &\quad \times \mathbb{E}\left[\left(1(x) - Rq_x\right)\left(1(x) - Rq_x\right)^\intercal\right]^{-1} \mathbb{E}\left(1(x) - Rq_x\right)^\intercal. \end{aligned}$$

Note that the returns on the trading strategy can be rewritten as a multilinear mapping of the Arrow-Debreu securities, by virtue of the AD securities spanning the filtration. Consequently, the linear projection recovers the expected return of the strategy

$$
\begin{aligned}
&\text{Cov}\ \left(R^* S'/S - R, 1_x\right) \mathbb{E}\left[\left(1\left(x\right) - Rq_x\right)\left(1\left(x\right) - Rq_x\right)^\intercal\right]^{-1} \mathbb{E}\left(1\left(x\right) - Rq_x\right)^\intercal \\
=&\mathbb{E}\left[R^* S'/S - R\right]
\end{aligned}
$$

as $x \to \mathbb{E}[x]$. Thus recovering the Arrow-Pratt portfolio solution $\theta^\intercal = RRA^{-1}\Sigma\mathbb{E}\left[R^*S'/S - R\right]$ as intended.

Now, to finalize the proof, recall that the buyer's first order condition states that

$$
\mathbb{E}\left[U' W_0 \left(R^* S'/S - R\right)\right] = 0.
$$

Using $U'W_0$ as $M^j$ it follows that $\mathbb{E}\left[\frac{M^j}{\mathbb{E}[M^j]} R^* S'/S\right] = R$. Meanwhile, because the seller is risk neutral we know that $\mathbb{E}\left[R^* S'/S\right] = \mathbb{E}\left[\frac{M^i}{\mathbb{E}[M^i]} R^* S'/S\right]$. Combining, we get

$$
\begin{aligned}
\mathbb{E}\left[R^* S'/S - R\right] &= \mathbb{E}\left[\left(\frac{M^i}{\mathbb{E}\left[M^i\right]} - \frac{M^j}{\mathbb{E}\left[M^j\right]}\right) R^* S'/S\right] \\
&= R\Delta_{ij}\tau.
\end{aligned}
$$

Rewriting, $\theta^\intercal = RRA^{-1}\Sigma^{-1}R\Delta_{ij}\tau$ so that $\Delta_{ij}\tau := R^{-1} \cdot RRA \cdot \Sigma \cdot \theta$.

## A.7 Proof of Proposition 9

The firm now jointly maximizes

$$
\max_{P_0, \beta^\intercal, B, I} \mathbb{E}\left[M^i \left(\underbrace{\pi\left(P, Q, I, x\right)}_{\text{Gross Income}} - \underbrace{R^*\left(B^*\right) \frac{S'}{S} B^*}_{\text{Foreign CC}} - \underbrace{R\left(B\right) B}_{\text{Domestic CC}}\right)\right]
$$

such that

$$
B + B^* \geq I
$$

give the same buyer IC, buyer PC, and nominal rigidity constraints. Rewrite the problem in terms of the risk neutral measure. The first order conditions of the problem are identical for the price, denomination, and quantity margins. It now features the bond margin satisfying

$$
\begin{aligned}
\mathbb{E}\left[M^i \left(\pi_I - \partial_{B^*} R^* \frac{S'}{S} B - R^* \frac{S'}{S}\right)\right] &= 0 \\
\mathbb{E}\left[M^i \left(\pi_I - \partial_B R B - R\right)\right] &= 0.
\end{aligned}
$$

Differencing the two conditions, we get

$$\mathbb{E}\left[M^i\left(\left(1+\partial_{b^*}r^*\right)R^*\frac{S'}{S}-\left(1+\partial_b r\right)R\right)\right]=0\Rightarrow\frac{1+\tau^{i*}}{1+\tau^i}=\frac{1+\partial_b r}{1+\partial_{b^*}r^*}$$

using the fact that the seller has competitive preferences, it follows that

$$\frac{1+\tau^{j*}}{1+\tau^j}\frac{R}{R^*}S=\mathbb{E}\left[\frac{M^j}{\mathbb{E}\left[M^j\right]}S'\right]=F.$$

Rearranging this equation

$$\frac{1+\tau^{j*}}{1+\tau^j}=\frac{F}{S}\frac{R^*}{R}$$

and combining with earlier

$$\Delta_{ij}\tau:=\frac{1+\partial_b r}{1+\partial_{b^*}r^*}-\frac{F}{S}\frac{R^*}{R}.$$

The rest of the problem remains identical since the investment margin is linearly separable.

## A.8 Proof of Proposition 10

Recall that the seller receives exchange rate risk and the buyer sells exchange rate risk. Consequently, the seller and buyer equate their Euler equations to the realized bid and ask price of $S'/S$. This corresponds to the following two equations,

$$\underbrace{1=\mathbb{E}\left[M^i\frac{S_{ic}^{bid'}}{S_{ic}^{ask}}R_c\right]}_{\text{Seller Long 1 FX}};\qquad\underbrace{1=\mathbb{E}\left[M^j\frac{S_{jc}^{ask'}}{S_{jc}^{bid}}R_c\right]}_{\text{Buyer Short 1 FX}}.$$

To derive the wedges $\tau^i$ and $\tau^j$, we need to rearrange the definitions in terms of the hypothetical market rate.

$$1=\mathbb{E}\left[M^i\frac{S_c'}{S_c}\frac{1+\tau_{ic}^{bid}}{1+\tau_{ic}^{ask}}R_c\right]=\mathbb{E}\left[M^j\frac{S_c'}{S_c}\frac{1+\tau_{jc}^{ask}}{1+\tau_{jc}^{bid}}R_c\right]$$

Since t-costs are constant across time

$$1+\tau^{i*}=\frac{1+\tau_i^{ask*}}{1+\tau_i^{bid*}};\qquad 1+\tau^{j*}=\frac{1+\tau_j^{bid*}}{1+\tau_j^{ask*}}$$

so

$$\Delta_{ij}\tau=\frac{\Delta_{bid/ask}\tau_i}{\Delta_{bid/ask}\tau_i^*}-\frac{\Delta_{bid/ask}\tau_j^*}{\Delta_{bid/ask}\tau_j}.$$

## A.9 Proof of Proposition 11

The seller's objective is given by

$$\max_{p_0,\beta,I,D}\int_X \left(\pi\left(P,Q,I,x\right)-D\right)1_{\pi\geq D}d\mu$$

subject to the constraints

$$\int_X\left[\pi\left(P,Q,I,\omega\right)-c\right]1_{\pi<D}d\mu+\int_X D1_{\pi\geq D}d\mu\geq I$$
$$\left(1-\delta\right)V^{-1}\left(P,x\right)+\delta\bar{V}^{-1}\left(\mathbb{E}\left[M^jP\right]\right)=Q$$

to solve this problem, we characterize the Lagrangian but instead differentiate on the CDF functions, which are implicit functions of the choice of prices and quantities.[22]

Assume the CDF for $x$ is given by $F$ and is continuously differentiable with pdf $f$. Define the CDF of the equilibrium profits as $G\left(\rho\right)$. Since $\pi$ is continuously differentiable, the pdf of profits $g\left(\rho\right)$ is well defined. Since $\pi\left(P\left(x\right),Q\left(x\right),x\right)$ is a function of the choice variable $P\left(x\right)$ and $Q\left(x\right)$, we can specify $\hat{G}\left(\rho;P\left(x\right),Q\left(x\right)\right)$ as the CDF taking into account the choice of price and quantity. Assume regularity conditions on the distribution $\lim_{L\to\infty}Lg\left(L\right)=0$ and $\lim_{L\to-\infty}G\left(L\right)=0$.

The maximization is equivalent to characterizing the Lagrangian,

$$\begin{aligned}\mathcal{L}=\min_{\eta(x_1),\eta,\lambda\geq0}\ \max_{p_0,\beta^{\intercal},q(\omega),D,I}&\int_D^{\infty}\left(\rho-D\right)d\hat{G}\left(\rho;P\left(x\right),Q\left(x\right)\right)\\&+\lambda\left(\int_{\infty}^{D}\left[\rho-c\right]d\hat{G}\left(\rho;P\left(x\right),Q\left(x\right)\right)+\int_D^{\infty}Dd\hat{G}\left(\rho;P\left(x\right),Q\left(x\right)\right)-I\right)\\&+\int\eta\left(x\right)\left[\left(1-\delta\right)V^{-1}\left(P,x\right)+\delta\bar{V}^{-1}\left(\mathbb{E}\left[P\right]\right)-Q\left(x\right)\right]dF\left(x\right)\end{aligned}$$

Specifically, the CDF $\hat{G}$ is defined as

$$\begin{aligned}\hat{G}\left(\rho;P\left(x\right),Q\left(x\right)\right)&=\mathbb{E}\left[1_{\pi(P(x),Q(x),x)\leq\rho}\right]\\&=\mathbb{E}\left[1_{\pi^*+\int_{Q^*(x)}^{Q(x)}\int_{P^*(x)}^{P(x)}\partial_{PQ}\pi(P,Q,x)dPdQ\leq\rho}\right]\\&=\int 1_{\pi^*(x)\leq\rho-\int_{Q^*(x)}^{Q(x)}\int_{P^*(x)}^{P(x)}\partial_{PQ}\pi(P,Q,x)dPdQ}dF\left(x\right)\\&=\int G\left(\rho-\int_{Q^*(x)}^{Q(x)}\int_{P^*(x)}^{P(x)}\partial_{PQ}\pi\left(P,Q,x\right)dPdQ\mid x\right)dF\left(x\right)\end{aligned}$$

[22]Special thanks to Sebastian Bauer for help on this proof.

thus differentiating

$$\begin{aligned}\partial_{Q(x)} d\hat{G}\left(\rho; P^*\left(x\right), Q^*\left(x\right)\right) &= -\int g'\left(\rho \mid x\right) \partial_Q \pi\left(P^*, Q^*, x\right) 1_x dF\left(x\right) \\ &= -g'\left(\rho \mid x\right) \partial_Q \pi\left(P^*, Q^*, x\right) dF\left(x\right)\end{aligned}$$

moreover, differentiating by $p_0$

$$\begin{aligned}\partial_{P_0} d\hat{G}\left(\rho; P^*\left(x\right), Q^*\left(x\right)\right) &= -\int g'\left(\rho \mid x\right) \partial_P \pi\left(P^*, Q^*, x\right) dF\left(x\right) \\ &= -\mathbb{E}\left[g'\left(\rho \mid x\right) \partial_P \pi\left(P^*, Q^*, x\right)\right]\end{aligned}$$

and

$$\partial_\beta d\hat{G}\left(\rho; P^*\left(x\right), Q^*\left(x\right)\right) = -\mathbb{E}\left[g'\left(\rho \mid x\right) \partial_P \pi\left(P^*, Q^*, x\right) s\right].$$

Using this result, we can now characterize the first order conditions as if they were directly differentiating on the CDF. Differentiating over $p_0$

$$\begin{aligned}0 = &-\int_D^\infty \int_X \left(\rho - D\right) g'\left(\rho \mid x\right) \partial_P \pi^*\left(x\right) dF\left(x\right) d\rho \\ &- \lambda\left(\int_{-\infty}^D \int_X \left[\rho - c\right] g'\left(\rho \mid x\right) \partial_P \pi^*\left(x\right) dF\left(x\right) d\rho + \int_D^\infty \int_X D g'\left(\rho \mid x\right) \partial_P \pi^*\left(x\right) dF\left(x\right) d\rho\right) \\ &+ \int_X \eta\left(x\right) \left[\left(1 - \delta\right) \partial_P V^{-1}\left(P, x\right) + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}\left(\mathbb{E}\left[P\right]\right)\right] dF\left(x\right)\end{aligned}$$

since our functions are bounded, we can apply Fubini's Theorem and tuck the integral over $D$ inside. Integrating by parts,

$$\begin{aligned}\int_D^\infty \left(\rho - D\right) g'\left(\rho \mid x\right) d\rho &= \lim_{L \to \infty} L g\left(L \mid x\right) - D g\left(D \mid x\right) - \int_D^\infty g\left(\rho \mid x\right) d\rho - \int_D^\infty D g'\left(\rho \mid x\right) d\rho \\ &= -D g\left(D \mid x\right) - \left(G\left(\infty \mid x\right) - G\left(D \mid x\right)\right) - D\left[g\left(\infty \mid x\right) - g\left(D \mid x\right)\right] \\ &= -\left(1 - G\left(D \mid x\right)\right)\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^D \left[\rho - c\right] g'\left(\rho \mid x\right) d\rho &= \left(D - c\right) g\left(D \mid x\right) - G\left(D \mid x\right) \\ \int_D^\infty D g'\left(\rho \mid x\right) d\rho &= D\left(g\left(\infty \mid x\right) - g\left(D \mid x\right)\right) = -D g\left(D \mid x\right)\end{aligned}$$

Using these conditions we get

$$
\begin{aligned}
0 &= \int_X \left[1 - G\left(D \mid x\right)\right] \partial_P \pi^* \left(x\right) dF\left(x\right) \\
&\quad + \lambda \left( \int_X \left[G\left(D \mid x\right) - \left(D - c\right) g\left(D \mid x\right)\right] \partial_P \pi^* dF\left(x\right) + \int_X D g\left(D \mid x\right) \partial_P \pi^* dF\left(x\right) \right) \\
&\quad + \int_X \eta\left(x\right) \Big[\left(1 - \delta\right) \partial_P V^{-1}\left(P, x\right) + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}\left(\mathbb{E}\left[P\right]\right)\Big] dF\left(x\right) \\
&= \int_X \left[1 - \left(1 - \lambda\right) G\left(D \mid x\right) + c g\left(D \mid x\right)\right] \partial_P \pi^* \left(x\right) dF\left(x\right) \\
&\quad + \int_X \eta\left(x\right) \Big[\left(1 - \delta\right) \partial_P V^{-1}\left(P, x\right) + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}\left(\mathbb{E}\left[P\right]\right)\Big] dF\left(x\right)
\end{aligned}
$$

similarly for $\beta$ we get the first order condition

$$
\begin{aligned}
0 = &- \int_D^\infty \int_X \rho g'\left(\rho \mid x\right) \partial_P \pi^* \left(x\right) s dF\left(x\right) d\rho \\
&- \lambda \int_0^D \int_X \left[\rho - c\right] g'\left(\rho \mid x\right) \partial_P \pi^* \left(x\right) s dF\left(x\right) d\rho \\
&- \lambda \int_D^\infty \int_X D g'\left(\rho \mid x\right) \partial_P \pi^* \left(x\right) s dF\left(x\right) d\rho \\
&+ \int_X \eta\left(x\right) \Big[\left(1 - \delta\right) \partial_P V^{-1}\left(P, x\right) s + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}\left(\mathbb{E}\left[P\right]\right) \mathbb{E}\left[s\right]\Big] dF\left(x\right)
\end{aligned}
$$

since the $s$ term does not change the integration identities as they hold $x$ fixed, we get

$$
\begin{aligned}
0 = &\int_X \left[1 - \left(1 - \lambda\right) G\left(D \mid x\right) + c g\left(D \mid x\right)\right] \partial_P \pi^* \left(x\right) s dF\left(x\right) \\
&+ \int_X \eta\left(x\right) \Big[\left(1 - \delta\right) \partial_P V^{-1}\left(P, x\right) s + \delta \partial_{\mathbb{E}[P]} \bar{V}^{-1}\left(\mathbb{E}\left[P\right]\right) \mathbb{E}\left[s\right]\Big] dF\left(x\right).
\end{aligned}
$$

For the quantity margin, our first order condition is

$$
\begin{aligned}
0 = &- \int_D^\infty \rho g'\left(\rho \mid x\right) \partial_{Q(x)} \pi^* \left(x\right) dF\left(x\right) d\rho \\
&- \lambda \left( \int_0^D \left[\rho - c\right] g'\left(\rho \mid x\right) \partial_{Q(x)} \pi^* \left(x\right) dF\left(x\right) d\rho + \int_D^\infty D g'\left(\rho \mid x\right) \partial_{Q(x)} \pi^* \left(x\right) dF\left(x\right) d\rho \right) \\
&- \int_X \eta\left(x\right) 1_x dF\left(x\right)
\end{aligned}
$$

once again using the integration identities and dividing through by $\int 1_x dF\left(x\right)$, we get:

$$
0 = \left[1 - \left(1 - \lambda\right) G\left(D \mid x\right) + \lambda c g\left(D \mid x\right)\right] \partial_{Q(x)} \pi^* \left(x\right) - \eta\left(x\right)
$$

define $\frac{dF^i(x)}{dF} = 1 - \left(1 - \lambda^*\right) G\left(D^* \mid x\right) + \lambda^* c g\left(D^* \mid x\right)$ and $\frac{dF^j}{dF} = 1$ as the constant in front of the revenue choice thus rewriting

$$\mathbb{E}^{\mu^i}\left[\partial_P \pi^*\right] + \mathbb{E}\left[\eta\left(x\right)\left\{\left(1-\delta\right)\partial_P V^{-1} + \delta \partial_{\mathbb{E}[P]}\bar{V}^{-1}\right\}\right] = 0$$
$$\mathbb{E}^{\mu^i}\left[\partial_P \pi^* s\right] + \mathbb{E}\left[\eta\left(x\right)\left\{\left(1-\delta\right)\partial_P V^{-1}s + \delta \partial_{\mathbb{E}[P]}\bar{V}^{-1}\mathbb{E}\left[s\right]\right\}\right] = 0$$
$$\frac{dF^i}{dF}\partial_{Q(x)}\pi^* - \eta\left(x\right) = 0$$

This is equivalent to the first-order conditions of the full problem where $\frac{M^i/\mathbb{E}\left[M^i\right]}{M^j/\mathbb{E}\left[M^j\right]} = \frac{dF^i}{dF}$ and $M^j := 1$. Intuitively, $1 - \left(1-\lambda^*\right)G\left(D^* \mid x\right)$ captures the direct effect of changing profits when the firm is in the interior of the default region. The term $\lambda^* c g\left(D^* \mid x\right)$ captures the marginal effect of moving the profits nearer the no default region. When the profits marginally enter the no-default region, they reduce the probability that the investor needs to pay $c$.

To pin down $\frac{dF^i(x)}{dF}$ we now use the first order condition on the debt choice to recover $\lambda$.

$$0 = -\int_D^\infty dG\left(\rho\right) + \lambda\left[D-c\right]g\left(D\right) - \lambda D g\left(D\right) + \lambda\int_D^\infty dG\left(\rho\right)$$
$$\left(1-G\left(D\right)\right) = \lambda\left(1 - G\left(D\right) - \left(1-\delta\right)Dg\left(D\right)\right)$$

thus

$$\lambda^* = \frac{1-G\left(D^*\right)}{1-G\left(D^*\right)-cg\left(D^*\right)} = 1 + \frac{cg\left(D^*\right)}{1-G\left(D^*\right)-cg\left(D^*\right)}$$

thus

$$\begin{aligned}&1-\left(1-\lambda\right)G\left(D^* \mid x\right) + \lambda c g\left(D^* \mid x\right)\\ =&1 + \frac{\left(1-G\left(D^*\right)\right)cg\left(D^* \mid x\right) + cg\left(D\right)G\left(D \mid x\right)}{1-G\left(D^*\right)-cg\left(D^*\right)}\end{aligned}$$

Due to profits being entirely characterized by $x$, we have the identity $G\left(D \mid x\right) = 1_{\pi(x)\leq D}$. Consequently, $g\left(D^* \mid x\right)$ is formalized using the Dirac delta function at $D^*$ . From the risk neutrality of the buyer, it follows that

$$\begin{aligned}\mathbb{E}\left[\frac{M^i}{\mathbb{E}\left[M^i\right]}s\right] - \mathbb{E}\left[\frac{M^j}{\mathbb{E}\left[M^j\right]}s\right] &= \mathbb{E}\left[\left(\frac{\left(1-G\left(D^*\right)\right)cg\left(D^* \mid x\right) + cg\left(D\right)G\left(D \mid x\right)}{1-G\left(D^*\right)-cg\left(D^*\right)}\right)s\right]\\ &= \frac{cg\left(D^*\right)}{1-G\left(D^*\right)-cg\left(D^*\right)}\mathbb{E}\left[1_{\pi^*\leq D^*}s\right]\\ &\quad + \frac{1-G\left(D^*\right)}{1-G\left(D^*\right)-cg\left(D^*\right)}c\int_\Omega g\left(D^* \mid x\left(\omega\right)\right)s\left(\omega\right)d\mu\left(\omega\right)\end{aligned}$$

Integrating over the Dirac delta measure and using a change of measure,

$$\begin{aligned}\int_{\Omega} g\left(D^{*} \mid x\left(\omega\right)\right) s\left(\omega\right) d\mu\left(\omega\right) &= \int_{-\infty}^{\infty} \int_{X} g\left(D^{*} \mid \rho\right) s dF\left(x \mid \rho\right) dG\left(\rho\right) \\ &= \mathbb{E}\left[s \mid \pi^{*} = D^{*}\right] g\left(D^{*}\right)\end{aligned}$$

This gives us

$$\begin{aligned}\Delta_{ij}\tau &= \frac{\left(1 - G\left(D^{*}\right)\right) \mathbb{E}\left[s \mid \pi^{*} = D^{*}\right] + \mathbb{E}\left[1_{\pi^{*} \leq D^{*}} s\right]}{1 - G\left(D^{*}\right) - cg\left(D^{*}\right)} cg\left(D^{*}\right) \\ &= \frac{cg\left(D^{*}\right) s_{\pi \leq D}}{1 - G\left(D^{*}\right) - cg\left(D^{*}\right)}\end{aligned}$$

where $s_{\pi \leq D} = \left(1 - G\left(D^{*}\right)\right) \mathbb{E}\left[s \mid \pi^{*} = D^{*}\right] + G\left(D^{*}\right) \mathbb{E}\left[s \mid \pi^{*} \leq D^{*}\right]$ is the distress price.